

## Twisted Surfaces Family in Euclidean 3-Space

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**Abstract** – In this study, we examine a distinct set of twisted surfaces in the three-dimensional Euclidean space  $\mathbb{E}^3$ . Our focus lies in the investigation of the differential geometry of this surface family, including the determination of their curvatures. Furthermore, we establish the essential conditions for minimal surfaces within this framework. Additionally, we calculate the Laplace–Beltrami operator for this particular surface family and provide an illustrative example.

**Keywords** – Euclidean 3-Space, Twisted Surfaces Family, Gauss Map, Curvatures, Laplace–Beltrami Operator

### I. INTRODUCTION

Initially, Chen [4, 5, 6, 7] proposed the notion of sub-manifolds ( $sm$ ) of finite order, which are immersed in Euclidean space  $\mathbb{E}^m$  or pseudo-Euclidean space  $\mathbb{E}_\nu^m$ . This was accomplished by employing a limited set of eigenfunctions obtained from the Laplacian operator. Subsequently, significant attention and research efforts have been devoted to exploring and investigating this subject matter.

Takahashi established that a Euclidean submanifold is categorized as 1-type if and only if it is either minimal or minimal within a hypersphere of  $\mathbb{E}^m$ . The construction of minimal submanifolds was pioneered by Lawson [20]. Subsequently, Garay [16] investigated Takahashi's theorem in the context of  $\mathbb{E}^m$ . Aminov [2] extensively examined the geometric properties of  $sm$ . Chen et al. [8], throughout four decades, devoted their research endeavors to the study of 1-type submanifolds and the 1-type Gauss map ( $Gm$ ) within the framework of space forms.

In the three-dimensional Euclidean space, denoted as  $\mathbb{E}^3$ , Takahashi [22] conducted an

investigation into the properties of minimal surfaces. Ferrandez et al. [14] established that surfaces exhibiting specific characteristics are either minimal cross-sections of a sphere or a right circular cylinder. Choi and Kim [10] directed their research towards the study of the minimal helicoid, which demonstrates a pointwise 1-type Gauss map  $Gm$  of the first kind. Garay [15] introduced a class of surfaces of finite type based on revolution. Dillen et al. [11] explored a distinct set of surfaces characterized by certain properties, including minimal surfaces, spheres, and circular cylinders.

Moreover, significant research endeavors have been conducted by Berger and Gostiaux [3], Do Carmo [12], Gray [17], and Kreyszig [18] regarding the investigation of twisted surfaces named helicoids.

The purpose of this research is to investigate the properties of a family of twisted surfaces in the three-dimensional Euclidean space  $\mathbb{E}^3$ . Our specific objectives are to compute the matrices associated with the fundamental form,  $Gm$ , and shape operator ( $so$ ) for this surface family. Utilizing the Cayley–Hamilton theorem, our aim is to determine the curvatures of these surfaces. Moreover, we seek

to establish the conditions for determining minimality within this framework. Additionally, we aim to explore the connection between the Laplace–Beltrami operator and these specific types of surfaces.

Section 2 provides an extensive elucidation of the fundamental principles and concepts that form the basis of three-dimensional Euclidean geometry.

Section 3 is devoted to presenting the curvature formulas to surfaces in  $\mathbb{E}^3$ .

In Section 4, we provide a comprehensive definition of the family of twisted surfaces, highlighting their unique properties and characteristics.

Section 5 shifts the focus towards the discussion of the Laplace–Beltrami operator for a smooth function in  $\mathbb{E}^3$ , along with the application of the aforementioned surfaces in its computation.

Lastly, we conclude the research in the final section.

## II. PRELIMINARIES

In this work, we adopt the subsequent notations, formulas, equations (Eqs.), and other relevant expressions.

Let  $M$  be an oriented hypersurface in  $\mathbb{E}^{n+1}$  with its so  $S$ , position vector  $x$ . Consider a local orthonormal frame field  $\{e_1, e_2, \dots, e_n\}$  consisting of principal directions of  $M$  coinciding with the principal curvature  $k_i$  for  $i = 1, 2, \dots, n$ .

Let the dual basis of this frame field be  $\{\Omega_1, \Omega_2, \dots, \Omega_n\}$ . Then, the first structural Eq. of Cartan is determined by

$$d\Omega_i = \sum_{j=1}^n \Omega_j \wedge \omega_{ij},$$

where  $\omega_{ij}$  indicates the connection forms coinciding with the chosen frame field. By the Codazzi Eq., we derive the Eqs.:

$$\begin{aligned} e_i(k_j) &= \omega_{ij}(e_j)(k_i - k_j), \\ \omega_{ij}(e_l)(k_i - k_j) &= \omega_{il}(e_j)(k_i - k_l), \end{aligned}$$

for different  $i, j, l = 1, 2, \dots, n$ .

We assume

$$s_j = \sigma_j(k_1, k_2, \dots, k_n),$$

where  $\sigma_j$  denotes the  $j$ -th elementary symmetric function defined by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq n} a_{i_1} a_{i_2} \dots a_{i_j}.$$

We give the notation

$$\Phi_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, \dots, k_n).$$

We get

$$\Phi_i^0 = 1,$$

and

$$s_{n+1} = s_{n+2} = \dots = 0.$$

The  $s_k$  is referred to as the  $k$ -th mean curvature ( $\mathcal{MC}$ ) of the hypersurface  $M$ . The  $\mathcal{MC}$  is described by

$$H = \frac{1}{n} s_1,$$

and the Gauss–Kronecker curvature of  $M$  is determined by

$$K = s_n.$$

If  $s_j \equiv 0$ , the hypersurface  $M$  is known as  $j$ -minimal.

In Euclidean  $(n+1)$ -space, to obtain the curvature formulas  $\mathcal{K}_i$  (See [1] and [19] for details.),  $i = 0, 1, \dots, n$ , we have the following characteristic polynomial Eq.:

$$P_S(\chi) = \sum_{k=0}^n (-1)^k s_k \chi^{n-k} = 0,$$

that is,

$$\det(S - \chi)J_n = 0. \quad (2.1)$$

Here,  $i = 0, 1, \dots, n$ ,  $J_n$  denotes the identity matrix. Hence, we obtain the curvature formulas

$$\binom{n}{i} \mathcal{K}_i = s_i.$$

We consider the immersion  $\mathbf{r} = \mathbf{r}(u, v)$  from  $M^2 \subset \mathbb{E}^2$  to  $\mathbb{E}^3$ .

**Definition 1.** An inner product of two vectors

$$\mathbf{a} = (a^1, a^2, a^3) \text{ and } \mathbf{b} = (b^1, b^2, b^3)$$

of  $\mathbb{E}^3$  is determined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = a^1 b^1 + a^2 b^2 + a^3 b^3.$$

**Definition 2.** A vector product of

$$\mathbf{a} = (a^1, a^2, a^3) \text{ and } \mathbf{b} = (b^1, b^2, b^3)$$

of  $\mathbb{E}^3$  is defined by

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{pmatrix}.$$

**Definition 3.** The matrix

$$(\mathbf{g}_{ij})^{-1} (\mathbf{h}_{ij})$$

describes the so matrix  $S$  of surface  $\mathbf{r}$  in Euclidean 3-space  $\mathbb{E}^3$ , where

$$(\mathbf{g}_{ij})_{2 \times 2} \text{ and } (\mathbf{h}_{ij})_{2 \times 2}$$

determine the first and the second fundamental form matrices, respectively, and

$$\mathbf{g}_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j \rangle, \quad \mathbf{h}_{ij} = \langle \mathbf{r}_{ij}, \mathbb{G} \rangle, \quad i, j = 1, 2,$$

$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$  when  $i = 1$ ,  $\mathbf{r}_{uv} = \frac{\partial^2 \mathbf{r}}{\partial u \partial v}$  when  $i = 1$  and  $j = 2$ , etc.,  $e_k$  denotes the base elements of  $\mathbb{E}^3$ , and

$$\mathbb{G} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \quad (2.2)$$

indicates the Gm of the surface  $\mathbf{r}$ .

### III. CURVATURES IN THREE-SPACE

In this section, we give the curvature formulas of any surface  $\mathbf{r} = \mathbf{r}(u, v)$  in  $\mathbb{E}^3$ .

**Theorem 1.** A surface  $\mathbf{r}$  in  $\mathbb{E}^3$  holds the following formulas,

$$\mathcal{K}_0 = 1, \quad 2\mathcal{K}_1 = -\frac{\eta_1}{\eta_2}, \quad \mathcal{K}_2 = \frac{\eta_0}{\eta_2} \quad (3.1)$$

where

$$\eta_2 \tau^2 + \eta_1 \tau + \eta_0 = 0$$

determines the characteristic polynomial Eq. of the so matrix,

$$\eta_2 = \det(\mathbf{g}_{ij}), \quad \eta_0 = \det(\mathbf{h}_{ij}),$$

$(\mathbf{g}_{ij})_{2 \times 2}$  indicates the first fundamental form matrix and  $(\mathbf{h}_{ij})_{2 \times 2}$  denotes the second fundamental form matrix.

*Proof.* The matrix

$$(\mathbf{g}_{ij})^{-1} (\mathbf{h}_{ij})$$

determines the so matrix of surface  $\mathbf{r}$  in Euclidean 3-space  $\mathbb{E}^3$ . We obtain the characteristic polynomial Eq.

$$\det(S - \chi J_2) = 0.$$

Then, we have the curvatures

$$\binom{2}{0} \mathcal{K}_0 = 1,$$

$$\binom{2}{1} \mathcal{K}_1 = -\frac{\eta_1}{\eta_2},$$

$$\binom{2}{2} \mathcal{K}_2 = \frac{\eta_0}{\eta_2}.$$

**Definition 4.** A surface  $\mathbf{r}$  is named  $j$ -minimal if  $\mathcal{K}_j = 0$ , where  $j = 1, 2$ .

**Theorem 2.** A surface  $\mathbf{r} = \mathbf{r}(u, v)$  in  $\mathbb{E}^3$  has the relation

$$\mathcal{K}_0(\mathbf{t}_{ij}) - 2\mathcal{K}_1(\mathbf{h}_{ij}) + \mathcal{K}_2(\mathbf{g}_{ij}) = \mathcal{O}_2,$$

where  $(\mathbf{g}_{ij}), (\mathbf{h}_{ij}), (\mathbf{t}_{ij})$  determine the fundamental form matrices,  $\mathcal{O}_2$  represents the zero matrix.

*Proof.* Taking  $n = 2$  in (2.1), it works.

#### IV. TWISTED SURFACES FAMILY IN $\mathbb{E}^3$

In this section, we define the twisted surfaces family ( $\mathcal{TSF}$ ), then find its differential geometric properties in Euclidean 3-space  $\mathbb{E}^3$ .

A ruled surface, denoted by

$$\mathbf{r}(u, v) = p(v) + u q(v)$$

$$= a(0, 0, v)$$

$$+ u(\cos v, \sin v, 0)$$

where  $a \neq 0$ , can be classified as a right twisted (or helicoid) in  $\mathbb{E}^3$  if it can be generated by translating a straight line that intersects a fixed straight line, while maintaining a perpendicular relationship between the lines throughout the generation process. By considering the  $xy$ -plane as the perpendicular plane and selecting the  $z$ -axis as the reference line, the parametric equation for the right twisted surface is given as:

$$\mathbf{r}(u, v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \cos v \\ u \sin v \\ pv \end{pmatrix},$$

where  $u \in \mathbb{R}$ ,  $0 \leq v < 2\pi$ , and  $p \neq 0$  represents the pitch. Further details can be found in the works of Berger and Gostiaux [3], Do Carmo [12], Gray [17], and Kreyszig [18].

**Definition 5.** A  $\mathcal{TSF}$  is an immersion  $\mathbf{t}$  from  $M^2 \subset \mathbb{E}^2$  to  $\mathbb{E}^3$  with rotating axis  $z$ , defined by

$$\mathbf{t}(u, v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(u) \cos g(v) \\ f(u) \sin g(v) \\ h(u) + pk(v) \end{pmatrix} \quad (4.1)$$

where pitch  $p \neq 0$ , and

$$f = f(u), \quad g = g(v), \quad h = h(u), \quad k = k(v)$$

denote the differentiable functions.

Taking the first derivatives of  $\mathcal{TSF}$  determined by Eq. (4.1) w.r.t.  $u, v$ , respectively, we obtain the first fundamental form matrix

$$(g_{ij}) = \begin{pmatrix} f_u^2 + h_u^2 & pf_u k_v \\ pf_u k_v & f^2 g_v^2 + p^2 k_v^2 \end{pmatrix} \quad (4.2)$$

where

$$f_u^2 = \left(\frac{\partial f}{\partial u}\right)^2, \quad g_v^2 = \left(\frac{\partial g}{\partial v}\right)^2,$$

$$h_u^2 = \left(\frac{\partial h}{\partial u}\right)^2, \quad k_v^2 = \left(\frac{\partial k}{\partial v}\right)^2.$$

Hence,

$$\mathcal{D} = \det(g_{ij}) = f^2(f_u^2 + h_u^2)g_v^2 + p^2 f_u^2 k_v^2.$$

Using (2.2) we obtain the following  $Gm$  of the  $\mathcal{TSF}$  determined by Eq. (4.1):

$$\mathbb{G} = \frac{1}{\mathcal{D}^{1/2}} \begin{pmatrix} pf_u k_v \sin g - fh_u g_v \cos g \\ -pf_u k_v \cos g - fh_u g_v \sin g \\ fh_u g_v \end{pmatrix}. \quad (4.3)$$

By taking the second derivatives w.r.t.  $u, v$ , of  $\mathcal{TSF}$  described by Eq. (4.1), and by using the  $Gm$  given by Eq. (4.3), we find the components of the second fundamental form matrix

$$h_{11} = \frac{1}{\mathcal{D}^{1/2}} fh_u g_v (f_u h_{uu} - h_u f_{uu}),$$

$$h_{12} = -\frac{1}{\mathcal{D}^{1/2}} pf_u^2 g_v k_v = h_{21}, \quad (4.4)$$

$$h_{22} = -\frac{1}{\mathfrak{D}^{1/2}} \mathfrak{f} \left( \mathfrak{f} \mathfrak{g}_v^3 h_u + \mathfrak{p} \mathfrak{f}_u (\mathfrak{g}_v h_{vv} - k_v \mathfrak{g}_{vv}) \right),$$

$$h_{uu} = \frac{\partial^2 h}{\partial u^2}, \quad h_{vv} = \frac{\partial^2 h}{\partial v^2},$$

ect. By using (4.2) and (4.4), we compute the following so matrix

and

$$\mathfrak{f}_{uu} = \frac{\partial^2 \mathfrak{f}}{\partial u^2}, \quad \mathfrak{g}_{vv} = \frac{\partial^2 \mathfrak{g}}{\partial v^2},$$

$$S = (s_{ij})_{2 \times 2}$$

of (4.1) with components

$$s_{11} = \frac{\mathfrak{g}_v [\mathfrak{f}^3 \mathfrak{g}_v^2 (\mathfrak{f}_u h_{uu} - h_u \mathfrak{f}_{uu}) + \mathfrak{p}^2 h_v^2 (\mathfrak{f} (\mathfrak{f}_u h_{uu} - h_u \mathfrak{f}_{uu}) + \mathfrak{f}_u^2 h_u)]}{\mathfrak{D}^{3/2}},$$

$$s_{12} = -\frac{\mathfrak{p} h_v [\mathfrak{f}_u^2 \mathfrak{g}_v^3 (\mathfrak{f}^2 + h_u^2) + \mathfrak{p}^2 \mathfrak{f}_u^2 \mathfrak{g}_v h_v^2 + \mathfrak{p} \mathfrak{f} \mathfrak{f}_u (\mathfrak{g}_v h_u h_{vv} - h_u h_v \mathfrak{g}_{vv})]}{\mathfrak{D}^{3/2}},$$

$$s_{21} = \frac{\mathfrak{p} \mathfrak{g}_v h_v [\mathfrak{f} h_u (h_u \mathfrak{f}_{uu} - \mathfrak{f}_u h_{uu}) - \mathfrak{f}_u^2 (\mathfrak{f}_u^2 + h_u^2)]}{\mathfrak{D}^{3/2}},$$

$$s_{22} = -\frac{\mathfrak{p} \mathfrak{f} \mathfrak{f}_u h_v \left[ (\mathfrak{f}_u^2 + h_u^2) \mathfrak{g}_{vv} + \mathfrak{g}_v \left( \mathfrak{p} \mathfrak{f} \mathfrak{f}_u \mathfrak{f}_u^2 + h_u^2 h_{vv} + h_u \left( (\mathfrak{f}^2 \mathfrak{g}_v^2 + \mathfrak{p}^2 h_v^2) \mathfrak{f}_u^2 + \mathfrak{f}^2 \mathfrak{g}_v^2 h_u^2 \right) \right) \right]}{\mathfrak{D}^{3/2}},$$

Finally, using (3.1), with (4.2), (4.4), respectively, we find the curvatures of the  $\mathcal{TSF}$  defined by Eq. (4.1) as follows.

**Theorem 3.** Let  $\mathfrak{t}$  be a  $\mathcal{TSF}$  determined by Eq. (4.1) in  $\mathbb{E}^3$ .  $\mathfrak{t}$  contains the following curvatures

$$\mathcal{K}_0 = 1,$$

$$\begin{aligned} 2\mathcal{K}_1 = & \frac{1}{\mathfrak{D}^{3/2}} (\mathfrak{p}^2 \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v h_v^2 h_{uu} + \mathfrak{f}^3 \mathfrak{f}_u \mathfrak{g}_v^3 h_{uu} + 2\mathfrak{p}^2 \mathfrak{f}_u^2 \mathfrak{g}_v h_u h_v^2 + \mathfrak{f}^2 \mathfrak{g}_v^3 \mathfrak{f}_u^2 h_u \\ & - \mathfrak{p}^2 \mathfrak{f} \mathfrak{g}_v h_u h_v^2 \mathfrak{f}_{uu} - \mathfrak{f}^3 \mathfrak{g}_v^3 h_u \mathfrak{f}_{uu} + \mathfrak{f}^2 \mathfrak{g}_v^3 h_u^3 - \mathfrak{p} \mathfrak{f} \mathfrak{f}_u^3 h_v \mathfrak{g}_{vv} + \mathfrak{p} \mathfrak{f} \mathfrak{f}_u^3 \mathfrak{g}_v h_{vv} \\ & - \mathfrak{p} \mathfrak{f} \mathfrak{f}_u h_u^2 h_v \mathfrak{g}_{vv} + \mathfrak{p} \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v h_u^2 h_{vv}), \end{aligned}$$

$$\begin{aligned} \mathcal{K}_2 = & \frac{1}{\mathfrak{D}^2} \mathfrak{g}_v (\mathfrak{f}^2 h_u (-\mathfrak{f} \mathfrak{g}_v^3 h_u + \mathfrak{p} \mathfrak{f}_u h_v \mathfrak{g}_{vv} - \mathfrak{p} \mathfrak{f}_u \mathfrak{g}_v h_{vv}) \mathfrak{f}_{uu} - (\mathfrak{f}^2 (-\mathfrak{f} \mathfrak{g}_v^3 h_u + \mathfrak{p} \mathfrak{f}_u \mathfrak{g}_{vv} h_v \\ & - \mathfrak{p} \mathfrak{g}_v \mathfrak{f}_u h_{vv}) h_{uu} + \mathfrak{p}^2 \mathfrak{f}_u^3 \mathfrak{g}_v h_v^2) \mathfrak{f}_u). \end{aligned}$$

Here,  $\mathcal{K}_1$  represents the  $\mathcal{MC}$ ,  $\mathcal{K}_2$  denotes the Gaussian curvature.

*Proof.* By using the Cayley–Hamilton theorem, we

reveal the following characteristic polynomial Eq. of the so matrix of  $\mathcal{TSF}$  defined by Eq. (4.1):

$$\mathcal{K}_0\chi^2 - 2\mathcal{K}_1\chi + \mathcal{K}_2 = 0$$

where

$$\mathcal{K}_0 = 1,$$

$$2\mathcal{K}_1 = s_{11} + s_{22},$$

$$\mathcal{K}_2 = s_{11}s_{22} - s_{12}s_{21}.$$

The curvatures  $\mathcal{K}_i$  of  $\mathfrak{t}$  are obtained by the above Eqs.

**Corollary 1.** *Let  $\mathfrak{t}$  be a  $\mathcal{TSF}$  defined by Eq. (4.1) in  $\mathbb{E}^3$ .  $\mathfrak{t}$  is 1-minimal iff the following partial differential Eq. holds*

$$\begin{aligned} & p^2 f f_u g_v k_v^2 h_{uu} + f^3 f_u g_v^3 h_{uu} \\ & + 2p^2 f_u^2 g_v h_u k_v^2 + f^2 g_v^3 f_u^2 h_u \\ & - p^2 f g_v h_u k_v^2 f_{uu} - f^3 g_v^3 h_u f_{uu} \\ & + f^2 g_v^3 h_u^3 - p f f_u^3 k_v g_{vv} \\ & + p f f_u^3 g_v k_{vv} - p f f_u h_u^2 k_v g_{vv} \\ & + p f f_u g_v h_u^2 k_{vv} = 0, \end{aligned}$$

where  $\mathfrak{D} \neq 0$ .

**Corollary 2.** *Let  $\mathfrak{t}$  be a  $\mathcal{TSF}$  determined by Eq. (4.1) in  $\mathbb{E}^3$ .  $\mathfrak{t}$  is 2-minimal iff the following partial differential Eq. reveals*

$$\begin{aligned} & f^2 g_v h_u (-f g_v^3 h_u + p f_u k_v g_{vv} - p f_u g_v k_{vv}) f_{uu} \\ & - (f^2 g_v (-f g_v^3 h_u + p f_u g_{vv} k_v - p g_v f_u k_{vv}) h_{uu} \\ & + p^2 f_u^3 g_v k_v^2) f_u = 0. \end{aligned}$$

where  $\mathfrak{D} \neq 0$ .

## V. LAPLACE–BELTRAMI OPERATOR OF THE TWISTED SURFACES FAMILY IN $\mathbb{E}^3$

In this section, our focus is on the Laplace–Beltrami operator ( $\mathcal{LB}\sigma$ ) of a smooth function in  $\mathbb{E}^3$ . We will proceed to compute it utilizing the  $\mathcal{TSF}$ , which is defined by Eq. (4.1).

**Definition 6.** *The  $\mathcal{LB}\sigma$  of a smooth function  $\varphi = \varphi(x^1, x^2)$  in  $\mathcal{D} \subset \mathbb{R}^2$  of class  $C^2$  is the operator defined by*

$$\Delta\varphi = \frac{1}{\mathfrak{D}^{1/2}} \sum_{i,j=1}^2 \frac{\partial}{\partial x^i} \left( \mathfrak{D}^{1/2} g^{ij} \frac{\partial\varphi}{\partial x^j} \right), \quad (5.1)$$

where

$$(g^{ij}) = (g_{kl})^{-1}$$

and

$$\mathfrak{D} = \det(g_{ij}).$$

Therefore, the  $\mathcal{LB}\sigma$  of the  $\mathcal{TSF}$  given by Eq. (4.1) is determined by

$$\begin{aligned} \Delta\mathfrak{t} = & \frac{1}{\mathfrak{D}^{1/2}} \left[ \frac{\partial}{\partial u} \left( \mathfrak{D}^{1/2} g^{11} \frac{\partial\mathfrak{t}}{\partial u} \right) + \frac{\partial}{\partial u} \left( \mathfrak{D}^{1/2} g^{12} \frac{\partial\mathfrak{t}}{\partial v} \right) \right. \\ & \left. + \frac{\partial}{\partial v} \left( \mathfrak{D}^{1/2} g^{21} \frac{\partial\mathfrak{t}}{\partial u} \right) + \frac{\partial}{\partial v} \left( \mathfrak{D}^{1/2} g^{22} \frac{\partial\mathfrak{t}}{\partial v} \right) \right], \quad (5.2) \end{aligned}$$

where

$$\begin{aligned} g^{11} &= \frac{f^2 g_v^2 + p^2 k_v^2}{\mathfrak{D}}, \\ g^{12} &= -\frac{p h_u k_v}{\mathfrak{D}} = g^{21}, \\ g^{22} &= \frac{f_u^2 + h_u^2}{\mathfrak{D}}. \end{aligned} \quad (5.3)$$

Taking the derivatives of the functions determined by Eqs. (5.3) in (5.2), w.r.t.  $u$  and  $v$ , resp., we find the following.

**Theorem 4.** *The  $\mathcal{LB}\sigma$  of the  $\mathcal{TSF}$   $\mathfrak{t}$  denoted by Eq. (4.1) is determined by*

*Proof.* With direct calculating by (5.2), we obtain

$$\Delta \mathfrak{t} = 2\mathcal{K}_1 \mathbb{G},$$

$$\Delta \mathfrak{t} = (\Delta \mathfrak{t}_1, \Delta \mathfrak{t}_2, \Delta \mathfrak{t}_3),$$

where  $\mathcal{K}_1$  describes the MC,  $\mathbb{G}$  represents the  $\mathbb{G}_m$  of  $\mathfrak{t}$ .

with components

$$\begin{aligned} \Delta \mathfrak{t}_1 = \frac{1}{\mathfrak{D}^2} & (p^2 \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v \mathfrak{h}_v^2 \mathfrak{h}_{uu} + \mathfrak{f}^3 \mathfrak{f}_u \mathfrak{g}_v^3 \mathfrak{h}_{uu} + 2p^2 \mathfrak{f}_u^2 \mathfrak{g}_v \mathfrak{h}_u \mathfrak{h}_v^2 + \mathfrak{f}^2 \mathfrak{g}_v^3 \mathfrak{f}_u^2 \mathfrak{h}_u \\ & - p^2 \mathfrak{f} \mathfrak{g}_v \mathfrak{h}_u \mathfrak{h}_v^2 \mathfrak{f}_{uu} - \mathfrak{f}^3 \mathfrak{g}_v^3 \mathfrak{h}_u \mathfrak{f}_{uu} + \mathfrak{f}^2 \mathfrak{g}_v^3 \mathfrak{h}_u^3 - p \mathfrak{f} \mathfrak{f}_u^3 \mathfrak{h}_v \mathfrak{g}_{vv} + p \mathfrak{f} \mathfrak{f}_u^3 \mathfrak{g}_v \mathfrak{h}_{vv} \\ & - p \mathfrak{f} \mathfrak{f}_u \mathfrak{h}_u^2 \mathfrak{h}_v \mathfrak{g}_{vv} + p \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v \mathfrak{h}_u^2 \mathfrak{h}_{vv}) (p \mathfrak{f}_u \mathfrak{h}_v \sin \mathfrak{g} - \mathfrak{f} \mathfrak{h}_u \mathfrak{g}_v \cos \mathfrak{g}), \end{aligned}$$

$$\begin{aligned} \Delta \mathfrak{t}_2 = \frac{1}{\mathfrak{D}^2} & (p^2 \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v \mathfrak{h}_v^2 \mathfrak{h}_{uu} + \mathfrak{f}^3 \mathfrak{f}_u \mathfrak{g}_v^3 \mathfrak{h}_{uu} + 2p^2 \mathfrak{f}_u^2 \mathfrak{g}_v \mathfrak{h}_u \mathfrak{h}_v^2 + \mathfrak{f}^2 \mathfrak{g}_v^3 \mathfrak{f}_u^2 \mathfrak{h}_u \\ & - p^2 \mathfrak{f} \mathfrak{g}_v \mathfrak{h}_u \mathfrak{h}_v^2 \mathfrak{f}_{uu} - \mathfrak{f}^3 \mathfrak{g}_v^3 \mathfrak{h}_u \mathfrak{f}_{uu} + \mathfrak{f}^2 \mathfrak{g}_v^3 \mathfrak{h}_u^3 - p \mathfrak{f} \mathfrak{f}_u^3 \mathfrak{h}_v \mathfrak{g}_{vv} + p \mathfrak{f} \mathfrak{f}_u^3 \mathfrak{g}_v \mathfrak{h}_{vv} \\ & - p \mathfrak{f} \mathfrak{f}_u \mathfrak{h}_u^2 \mathfrak{h}_v \mathfrak{g}_{vv} + p \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v \mathfrak{h}_u^2 \mathfrak{h}_{vv}) (-p \mathfrak{f}_u \mathfrak{h}_v \cos \mathfrak{g} - \mathfrak{f} \mathfrak{h}_u \mathfrak{g}_v \sin \mathfrak{g}), \end{aligned}$$

$$\begin{aligned} \Delta \mathfrak{t}_3 = \frac{1}{\mathfrak{D}^2} & (p^2 \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v \mathfrak{h}_v^2 \mathfrak{h}_{uu} + \mathfrak{f}^3 \mathfrak{f}_u \mathfrak{g}_v^3 \mathfrak{h}_{uu} + 2p^2 \mathfrak{f}_u^2 \mathfrak{g}_v \mathfrak{h}_u \mathfrak{h}_v^2 + \mathfrak{f}^2 \mathfrak{g}_v^3 \mathfrak{f}_u^2 \mathfrak{h}_u \\ & - p^2 \mathfrak{f} \mathfrak{g}_v \mathfrak{h}_u \mathfrak{h}_v^2 \mathfrak{f}_{uu} - \mathfrak{f}^3 \mathfrak{g}_v^3 \mathfrak{h}_u \mathfrak{f}_{uu} + \mathfrak{f}^2 \mathfrak{g}_v^3 \mathfrak{h}_u^3 - p \mathfrak{f} \mathfrak{f}_u^3 \mathfrak{h}_v \mathfrak{g}_{vv} + p \mathfrak{f} \mathfrak{f}_u^3 \mathfrak{g}_v \mathfrak{h}_{vv} \\ & - p \mathfrak{f} \mathfrak{f}_u \mathfrak{h}_u^2 \mathfrak{h}_v \mathfrak{g}_{vv} + p \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v \mathfrak{h}_u^2 \mathfrak{h}_{vv}) \mathfrak{f} \mathfrak{f}_u \mathfrak{g}_v, \end{aligned}$$

**Definition 7.** The surface  $\mathfrak{t}$  is called harmonic if each componets of  $\Delta \mathfrak{t}$  is zero.

The curvatures are determined by

**Example 1.** Substituting  $p = 1$ ,

$$\mathcal{K}_0 = 1, \mathcal{K}_1 = \frac{u^2 + 1}{(2u^2 + 1)^{3/2}}, \mathcal{K}_2 = -\frac{1}{(2u^2 + 1)^2}.$$

$$\mathfrak{f}(u) = u, \mathfrak{g}(v) = v, \mathfrak{h}(u) = u, \mathfrak{h}(v) = v$$

Then, we obtain

into a TSF defined by Eq. (4.1) in  $\mathbb{E}^3$ , we have the  $\mathbb{G}_m$  and the so matrix, respectively,

$$\Delta \mathfrak{t} = \frac{u^2 + 1}{(2u^2 + 1)^2} \begin{pmatrix} \sin v - u \cos v \\ -u \sin v - \cos v \\ u \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\mathbb{G} = \frac{1}{(2u^2 + 1)^{1/2}} \begin{pmatrix} \sin v - u \cos v \\ -u \sin v - \cos v \\ u \end{pmatrix},$$

$$\mathbb{S} = \begin{pmatrix} 0 & -\frac{1}{(2u^2 + 1)^{1/2}} \\ \frac{1}{(2u^2 + 1)^{1/2}} & \frac{u^2}{(2u^2 + 1)^{1/2}} \end{pmatrix}.$$

Finally, the surface is not minimal and non-harmonic.

**Example 2.** Substituting  $p = 1$ ,

$$\mathfrak{f}(u) = u, \mathfrak{g}(v) = v, \mathfrak{h}(u) = 0, \mathfrak{k}(v) = v$$

into a TSSF defined by Eq. (4.1) in  $\mathbb{E}^3$ , we obtain the  $\mathbb{G}m$  and the  $so$  matrix, respectively,

$$\mathbb{G} = \frac{1}{(u^2 + 1)^{1/2}} \begin{pmatrix} \sin v \\ -\cos v \\ u \end{pmatrix},$$

$$\mathbb{S} = \begin{pmatrix} 0 & -\frac{1}{(u^2 + 1)^{1/2}} \\ \frac{1}{(u^2 + 1)^{3/2}} & 0 \end{pmatrix}.$$

The curvatures are defined by

$$\mathcal{K}_0 = 1, \mathcal{K}_1 = 0, \mathcal{K}_2 = -\frac{1}{(u^2 + 1)^2}.$$

Hence, we get

$$\Delta t = (0,0,0).$$

Therefore, the surface is 1-minimal and harmonic.

## VI. CONCLUSIONS

This study is focused on the investigation of the geometric properties exhibited by the family of twisted surfaces within the three-dimensional Euclidean space.

The main objective is to analyze and gain a comprehensive understanding of the unique characteristics of these surfaces. The field of differential geometry plays a crucial role in providing crucial insights into the local geometry, including properties such as curvatures and tangent spaces, of the twisted surfaces family. The application of the Cayley–Hamilton theorem enables an effective determination of the curvatures of these specific surfaces by expressing the characteristic polynomial in terms of the corresponding matrices themselves.

Furthermore, this research establishes the necessary conditions for minimality within the conoid surfaces family, serving as criteria to identify when a surface can be considered minimal within this specific family. Additionally, the exploration of the Laplace–Beltrami operator sheds light on its

relationship with the twisted surfaces family.

Through this investigation, this research contributes to an enhanced understanding of the geometric properties, curvatures, minimality conditions, and the interplay with the Laplace–Beltrami operator within the family of twisted surfaces in the three-dimensional Euclidean space.

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