All Sciences Proceedings
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5th International Conference on Applied Engineering and Natural Sciences

July 10-12, 2023 : Konya, Turkey

https://www.icaens.com/

Investigation of Free Vibrations of Stepped Nanobeam Embedded In Elastic Foundation

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Abstract – The free vibration of stepped nanobeams embedded in the elastic foundation was investigated using Eringen's nonlocal elasticity theory. It is fixed at the system ends with a simple-simple support. The stepped nanbeam’s equations of motion are obtained by using Hamilton's principle. Multi-time scale, which is the perturbation methods, was used for the analytical solution of the equations. To observe the effects of nano size effect, elastic basis coefficient and step location, natural frequencies of the first three modes of the system were obtained for different non-local parameter values, elastic foundation coefficients, step rates and step positions. In the results, it was seen that the non-local parameter had a negative effect on the natural frequency. The elastic foundation coefficient has been shown to reduce vibration amplitudes.

Keywords –Stepped Nanobeam, Vibration, Nonlocal Elasticity, Elastic Foundation

I. INTRODUCTION

Nanotechnology, which we have started to feel its presence in all areas of the scientific world, has become one of the most important subjects of today's studies and many studies have been carried out [1]–[10]. When nano-structures are observed, some linear and superficial faults such as lattice gaps, steps or cracks in the material structure disrupt the continuity of the system. It is known that at the nanoscale, the properties of the material differ depending on its size. For this reason, not neglecting the physical properties in examining the mechanical behavior of nanomaterials such as mechanical vibrations will allow more accurate results in real engineering applications. In this context, there are many studies on beams with stepped and cracked surfaces [11]–[19]. There are very few studies on stepped nanobeams [18], [20]–[23].

In the literature, vibration behavior on stepped nanobeams under different conditions has been investigated using different theories and presented as summarized above. The vibrational behavior of the stepped nanobeam supported by an elastic foundation has not been studied so far. In this study, the vibration behavior of stepped nanobeam embedded in an elastic foundation is modeled using Eringen's nonlocal elasticity theory. The elastic foundation in which the stepped nanobeam is embedded is modeled based on linear spring foundations.

II. NONLOCAL ELASTICITY THEORY

Nanobeam’s material is defaulted to be a non-linear elastic material that conforms to a nonlocal elasticity theory. According to [24] Concepts, the constitutive equations for materials conforming to nonlocal elasticity be expressed as:

$$\sigma_{ij}^{n}(x^{*}) = \iiint_{(V)} a\left(\left|x^{*\prime} - x^{*}\right|, \tau\right)\sigma_{ij}^{c}(x^{*\prime})dV \quad (1)$$

Eq. (1), σ_{ij}^{n} denotes the tension tensor at non-local elasticity, σ_{ij}^{c} the classical (hooke) tension tensor, and V the volume. Here, a is the kernel function, which is assumed to express the effect of the stress state in $x^{*\prime} \in V$ and the stress-strain state in $x^{*} \in V$, and τ is the physical constant [24]. Different forms of kernel function $a(x^{*})$ in eq. (1) describe different approximate models of nonlocal elasticity. Suppose $a(x^{*})$ is a linear differential operator L function. In this case,

$$La\left(\left[x^{*\prime} - x^{*}\right]\right) = \delta\left(\left[x^{*\prime} - x^{*}\right]\right) \quad (2)$$

Here δ is Dirac's δ - function, have shown that the function can be obtained by taking a simple two-dimensional kernel function [24].

$$L(a) = (1 - (e_0 a)^2 \nabla^2) a(x^{*}) \quad (3)$$

Here ∇ is laplace operator. Eq. (3), e_0 is a physical constant. a is the repetitive interatomic distance parameter (lattice size) in the lattice structure of nanomaterials. Eringen named the $e_0 a$ expression as a small-scale parameter and suggested that its value should be taken in $e_0 a < 2$ nanometer scales [24]. According to Eqs. (1) - (3) the constitutive equation of nonlocal elasticity can be determined as follows,

$$\left(1 - e_0 a \nabla^2\right) \sigma_{ij}^{n} = \sigma_{ij}^{c} \quad (4)$$

For homogeneous isotropic Euler Bernoulli beam

$$\sigma(x^{*}) - (e_0 a)^2 \frac{\partial^2 \sigma(x^{*})}{\partial x^2} = E\varepsilon(x^{*}) \quad (5)$$

III. MATERIAL AND METHODS

Hamilton's principle was used to derive the equations of motion of the stepped nanobeam embedded in the elastic foundation. First, the Lagrangian of the system $£ = T - V$ was found. According to Hamilton's principle, the difference between the kinetic T and potential V energies of a system within the time integral variation must be zero. Here, the difference of kinetic and potential energies is defined as "Lagrangian (£)".

$$T = \frac{1}{2}\int_0^{x_s} \rho A_1 \left(\frac{\partial w_1^{*}}{\partial t^{*}}\right)^2 dx^{*} + \frac{1}{2}\int_{x_s}^{L} \rho A_2 \left(\frac{\partial w_2^{*}}{\partial t^{*}}\right)^2 dx^{*} \quad (6a)$$

$$\begin{aligned} V = {} & \frac{1}{2}\int_0^{x_s} \begin{pmatrix} EI_1 \dfrac{\partial^2 w_1^{*}}{\partial x^{*2}} + (e_0 a)^2 N \dfrac{\partial^2 w_1^{*}}{\partial x^{*2}} \\ -(e_0 a)^2 \rho A_1 \dfrac{\partial^2 w_1^{*}}{\partial t^{*2}} - (e_0 a)^2 k \end{pmatrix} \frac{\partial^2 w_1^{*}}{\partial x^{*2}} dx \\ & - \frac{1}{2}\int_0^{x_s} N \left(\frac{\partial w_1^{*}}{\partial x^{*}}\right)^2 + \frac{1}{2} k \int_0^{x_s} w_1^{*2} dx^{*} \\ & + \frac{1}{2}\int_{x_s}^{L} \begin{pmatrix} EI_2 \dfrac{\partial^2 w_2^{*}}{\partial x^{*2}} + (e_0 a)^2 N \dfrac{\partial^2 w_2^{*}}{\partial x^{*2}} \\ -(e_0 a)^2 \rho A_2 \dfrac{\partial^2 w_2^{*}}{\partial t^{*2}} - (e_0 a)^2 k \end{pmatrix} \frac{\partial^2 w_2^{*}}{\partial x^{*2}} dx \\ & - \frac{1}{2}\int_{x_s}^{L} N \left(\frac{\partial w_2^{*}}{\partial x^{*}}\right)^2 + \frac{1}{2} k \int_{x_s}^{L} w_2^{*2} dx^{*} \end{aligned} \quad (6b)$$

Here, ρ represents the density of the stepped nanobeam, A_1 and A_2 represent the cross-sectional areas of the stepped nanobeam. E is the modulus of elasticity of the nanobeam embedded in the stepped elastic foundation., I_1 and I_2 are the moment of inertias. L is defined as the length scale parameter of the stepped nanobeam, x_s is step place, k is elastic foundation stiffness, and N is the axial force. $(\ )^{*}$ represents dimensional parameters. The equations of motion and boundary conditions before and after the step of the stepped nano beam were found as follows, using Hamilton's:

$$\begin{aligned} & EI_1 \frac{\partial^4 w_1^{*}}{\partial x^{*4}} + \rho A_1 \left(\frac{\partial^2 w_1^{*}}{\partial t^{*2}} - (e_0 a)^2 \frac{\partial^4 w_1^{*}}{\partial t^{*2} \partial x^{*2}}\right) \\ & + k\left(w_1^{*} - (e_0 a)^2 \frac{\partial^2 w_1^{*}}{\partial x^{*2}}\right) \\ & = \frac{EA_1}{2\left[x_s + (L - x_s) / \left(\dfrac{r_2}{r_1}\right)^2\right]} \\ & \left[\int_0^{x_s} \left(\frac{\partial w_1^{*}}{\partial x^{*}}\right)^2 dx^{*} + \int_{x_s}^{L} \left(\frac{\partial w_2^{*}}{\partial x^{*}}\right)^2 dx^{*}\right] \\ & \left(\frac{\partial^2 w_1^{*}}{\partial x^{*2}} - (e_0 a)^2 \frac{\partial^4 w_1^{*}}{\partial x^{*4}}\right) \end{aligned} \quad (7)$$

$$EI_2 \frac{\partial^4 w_2^*}{\partial x^{*4}} + \rho A_2 \left(\frac{\partial^2 w_2^*}{\partial t^{*2}} - (e_0 a)^2 \frac{\partial^2 w_2^*}{\partial t^{*2} \partial x^{*2}} \right)$$
$$+k(w_2^* - (e_0 a)^2 \frac{\partial^2 w_2^*}{\partial x^{*2}})$$
$$= \frac{EA_1}{2\left[x_s + (L - x_s)/\left(\frac{r_2}{r_1}\right)^2 \right]} \tag{8}$$
$$\left[\int_0^{x_s} \left(\frac{\partial w_1^*}{\partial x^*} \right)^2 dx^* + \int_{x_s}^{L} \left(\frac{\partial w_2^*}{\partial x^*} \right)^2 dx^* \right]$$
$$\left(\frac{\partial^2 w_2^*}{\partial x^{*2}} - (e_0 a)^2 \frac{\partial^4 w_2^*}{\partial x^{*4}} \right)$$

For Simple-Simple Support,

$$\begin{aligned}
&\frac{\partial^2 w_1^*(0)}{\partial x^{*2}} = 0, \\
&\delta w_1^*(x_s) = \delta w_2^*(x_s), \\
&\delta w_1^*(0) = 0, \\
&\frac{\partial(\delta w_1^*(x_s))}{\partial x^*} = \frac{\partial(\delta w_2^*(x_s))}{\partial x^*}, \\
&\frac{\partial^2 w_2^*(L)}{\partial x^{*2}} = 0, \\
&-EI_1 \frac{\partial^2 w_1^*(x_s)}{\partial x^{*2}} + EI_2 \frac{\partial^2 w_2^*(x_s)}{\partial x^{*2}} = 0, \\
&\delta w_2^*(L) = 0 \\
&EI_1 \frac{\partial^3 w_1^*(x_s)}{\partial x^{*3}} - EI_2 \frac{\partial^3 w_2^*(x_s)}{\partial x^{*3}} = 0
\end{aligned} \tag{9}$$

Dimensionless parameters are associated with dimensional values marked with an asterisk and equations are nondimensionalized

$$\begin{aligned}
&x = \frac{x^*}{L}, w_{1,2} = \frac{w_{1,2}^*}{R_{1,2}}, \ t = \beta t^*, \ \gamma = \frac{e_0 a}{L}, \\
&\alpha = \frac{r_2}{r_1}, \ \eta = \frac{x_s}{L}, \beta = \frac{1}{L^2}\sqrt{\frac{EI_1}{\rho A_1}}, \Lambda = \frac{kL^4}{EI_1}
\end{aligned} \tag{10}$$

α is a dimensionless parameter that indicates the ratio of the radio of the steps at eq. (10). γ is a dimensionless non-local parameter. η is a dimensionless parameter expressing the step location. R is the parameter expressing the radius of inertia of the circular cross section stepped beam.

IV. PERTURBATION ANALYSIS

In this section, the approximate solution is obtained by the perturbation method. The multi-scale method, which is the perturbation methods, is applied for the solution [25]. The following expansion can be suggested for the displacement functions.

$$y_1(x,t:\varepsilon) = \varepsilon^0 y_{10}(x,T_0,T_1) + \varepsilon y_{11}(x,T_0,T_1) \tag{11}$$

$$y_2(x,t:\varepsilon) = \varepsilon^0 y_{20}(x,T_0,T_1) + \varepsilon y_{21}(x,T_0,T_1) \tag{12}$$

ε is a small parameter used in calculations. $T_0 = \varepsilon^0 t$ is a fast time scale, $T_1 = \varepsilon t$ is slow time scale. According to the time derivative expressions are written in terms of new time variables,

$$\begin{aligned}
&\partial/\partial t = D_0 + \varepsilon D_1 \\
&\partial^2/\partial t^2 = D_0^{\,2} + 2\varepsilon D_0 D_1
\end{aligned} \quad \text{Where, } D_n = \partial/\partial T \tag{13}$$

After expansion, the first and second terms of the expansion are separated as follows:

Order (ε^0)

$$y_{10}^{iv} + D_0^2 y_{10} - \gamma^2 D_0^2 y_{10}'' - \Lambda(y_{10} - \gamma^2 y_{10}'') = 0 \tag{14}$$

$$y_{20}^{iv} + \frac{1}{\alpha^2} D_0^2 y_{20} - \frac{\gamma^2}{\alpha^2} D_0^2 y_{20}'' - \Lambda(y_{20} - \gamma^2 y_{20}'') = 0 \tag{15}$$

Order (ε)

$$\begin{aligned}
&y_{11}^{iv} + D_0^{\,2} y_{11} + 2D_0 D_1 y_{10} - 2\gamma^2 D_0 D_1 y_{10}'' - \gamma^2 D_0^{\,2} y_{11}'' \\
&-\Lambda(y_{11} - \gamma^2 y_{11}'') = \Gamma_1 \left[\int_0^{\eta} (y_{10}'^2) dx + \alpha^2 \int_{\eta}^{1} (y_{20}'^2) dx \right] y_{10}'' \\
&-\Gamma_1 \gamma^2 \left[\int_0^{\eta} (y_{10}'^2) dx + \alpha^2 \int_{\eta}^{1} (y_{20}'^2) dx \right] y_{10}^{iv} \\
&+ F \cos \Omega t - 2\mu D_0 y_{10}
\end{aligned} \tag{16}$$

$$
\begin{aligned}
&y_{21}^{iv} + \frac{1}{\alpha^2} D_0^{\ 2} y_{21} + \frac{2}{\alpha^2} D_0 D_1 y_{20} - 2\frac{\gamma^2}{\alpha^2} D_0 D_1 y_{20}'' \\
&-\frac{\gamma^2}{\alpha^2} D_0^{\ 2} y_{21}'' - \Lambda(y_{21} - \gamma^2 y_{21}'') \\
&= \Gamma_2 \left[\int_0^\eta (y_{10}'^2) dx + \alpha^2 \int_\eta^1 (y_{20}'^2) dx \right] y_{20}'' \\
&- \Gamma_2 \gamma^2 \left[\int_0^\eta (y_{10}'^2) dx + \alpha^2 \int_\eta^1 (y_{20}'^2) dx \right] y_{20}^{iv} + \\
&F \cos \Omega t - 2\mu D_0 y_{20}
\end{aligned} \tag{17}
$$

Where,

$$
\Gamma_1 = \frac{1}{2\left(\eta + \frac{(1-\eta)}{\alpha^2}\right)}, \text{ and } \quad \Gamma_2 = \frac{1}{2\alpha^4\left(\eta + \frac{(1-\eta)}{\alpha^2}\right)}
$$

The equations in the ε^0 Order give the linear equation of motion and the linear frequency equation of the system. The equations in ε order show the effects coming from the nonlinear part. The boundary conditions can be represented as

$$
\begin{aligned}
&y_{10}(0) = 0, && y_{20}(1) = 0 \\
&y_{11}(\eta) = \alpha y_{21}(\eta), && y_{11}'(\eta) = \alpha y_{21}'(\eta) \\
&y_{11}''(0) = 0, && y_{21}''(1) = 0 \\
&y_{11}''(\eta) = \alpha^5 y_{21}''(\eta), && y_{11}'''(\eta) = \alpha^5 y_{21}'''(\eta)
\end{aligned} \tag{18}
$$

V. LINEER PROBLEM

The first perturbation order ε^0 is given in Eqs. (14) and (15); The solution can be represented as

$$
y_{10}(x, T_0, T_1) = A_1(T_1) e^{i\omega T_0} Y_1(x) + \bar{A}_1(T_1) e^{-i\omega T_0} \bar{Y}_1(x) \tag{19}
$$

$$
y_{20}(x, T_0, T_1) = A_2(T_1) e^{i\omega T_0} Y_2(x) + \bar{A}_2(T_1) e^{-i\omega T_0} \bar{Y}_2(x) \tag{20}
$$

If eqs. (19) and (20) are applied to eqs. (14) and (15),

$$
Y_1^{iv}(x) + (\omega^2 - \Lambda)\gamma^2 Y_1''(x) + (\Lambda - \omega^2) Y_1(x) = 0 \tag{21}
$$

$$
\begin{aligned}
&Y_2^{iv}(x) - \frac{1}{\alpha^2}\omega^2 Y_2(x) + \frac{\gamma^2}{\alpha^2}(\omega^2 - \Lambda) Y_2''(x) \\
&+ \frac{1}{\alpha^2}(\Lambda - \omega^2) Y_2(x) = 0
\end{aligned} \tag{22}
$$

Eqs. (23) and (24) can be used to solve Eqs. (21) and (22)

$$
\begin{aligned}
Y_1(x) &= c_{11} e^{ir_{11}x} + c_{12} e^{ir_{12}x} + c_{13} e^{ir_{13}x} + c_{14} e^{ir_{14}x} \\
&= c_{11}\left(e^{ir_{11}x} + \frac{c_{12}}{c_{11}} e^{ir_{12}x} + \frac{c_{13}}{c_{11}} e^{ir_{13}x} + \frac{c_{14}}{c_{11}} e^{ir_{14}x} \right)
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
Y_2(x) &= c_{21} e^{ikr_{21}x} + c_{22} e^{ikr_{22}x} + c_{23} e^{ikr_{23}x} + c_{24} e^{ikr_{24}x} \\
&= c_{21}\left(e^{ikr_{21}x} + \frac{c_{22}}{c_{21}} e^{ikr_{22}x} + \frac{c_{23}}{c_{21}} e^{ikr_{23}x} + \frac{c_{24}}{c_{21}} e^{ikr_{24}x} \right)
\end{aligned} \tag{24}
$$

Where, $k = \frac{1}{\sqrt{\alpha}}$

the scattering equations are obtained.

$$
\begin{aligned}
&r_{1n}^4 - (\gamma^2\omega^2 - \gamma^2\Lambda) r_{1n}^2 + (\Lambda - \omega^2) = 0 \\
&n = 1, 2, 3, 4
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&r_{2n}^4 k^4 - \frac{\gamma^2}{\alpha^2} k^2 (\omega^2 - \Lambda) r_{2n}^2 \\
&+ \frac{1}{\alpha^2} k^2 (\Lambda - \omega^2) = 0
\end{aligned} \tag{26}
$$

r_n roots can be obtained numerically after all the constant data are entered numerically. At this step, to see the boundary conditions effects in the linear problem, a coefficient matrix is created by substituting the boundary conditions in equations (25) and (26). The values that make the determinant calculation of the matrix given above zero give the natural frequencies of the system.

VI. RESULTS AND DISCUSSIONS

In the results section of the study, first of all, the nanoscale effect of the system was investigated by using different non-local parameters values $\gamma = 0.1 - 0.2 - 0.3 - 0.4 - 0.5$ in the first column of Table 1.

Table 1 The first three mode natural frequency values of the stepped nanobeam for different non-local parameter values and elastic foundation coefficients

	$\alpha = 0.5$	$\eta = 0.5$				$\alpha = 1.5$	$\eta = 0.5$		
	$\Lambda = 10$	$\Lambda = 100$	$\Lambda = 250$	$\Lambda = 500$		$\Lambda = 10$	$\Lambda = 100$	$\Lambda = 250$	$\Lambda = 500$
	$\gamma = 0.1$					$\gamma = 0.1$			
ω_1	5,07959	10,76110	16,30340	22,71130	ω_1	11,47520	14,88890	19,27900	24,93350
ω_2	20,69500	22,76580	25,85120	30,30320	ω_2	45,46080	46,44010	48,02790	50,56360
ω_3	40,50870	41,60480	43,37000	46,16230	ω_3	80,04560	80,60590	81,53100	83,05000
	$\gamma = 0.2$					$\gamma = 0.2$			
ω_1	4,89488	10,67520	16,24680	22,67070	ω_1	10,25300	13,96860	18,57750	24,39510
ω_2	16,27690	18,83980	22,47080	27,47610	ω_2	32,29630	33,66080	35,81970	39,15420
ω_3	27,94410	29,51050	31,95110	35,64930	ω_3	50,35890	51,24470	52,68790	55,00930
	$\gamma = 0.3$					$\gamma = 0.3$			
ω_1	4,65789	10,56860	16,17700	22,62070	ω_1	8,93411	13,03140	17,88350	23,87090
ω_2	12,81140	15,94150	20,10300	25,57600	ω_2	23,91290	25,72600	28,49250	32,58570
ω_3	20,37850	22,47850	25,59850	30,08790	ω_3	35,52560	36,77050	38,75660	41,85780
	$\gamma = 0.4$					$\gamma = 0.4$			
ω_1	4,42019	10,46600	16,11020	22,57300	ω_1	7,80426	12,28440	17,34670	23,47140
ω_2	10,44200	14,10800	18,68250	24,47520	ω_2	18,75600	21,01880	24,32670	29,01360
ω_3	15,85080	18,47290	22,16410	27,22590	ω_3	27,24420	28,84870	31,34080	35,10340
	$\gamma = 0.5$					$\gamma = 0.5$			
ω_1	4,20874	10,37850	16,05350	22,53250	ω_1	6,91165	11,73760	16,96380	23,18990
ω_2	8,81829	12,95230	17,82590	23,82780	ω_2	15,39350	18,08200	21,83940	26,96220
ω_3	12,94940	16,05260	20,19120	25,64540	ω_3	22,06670	24,01960	26,96180	31,25600

Table 1 values in first column were created under the conditions of step ratio $\alpha = 0.5$ and step location $\eta = 0.5$. In the second column the nanoscale effect was examined under the conditions of step ratio $\alpha = 1.5$ and step location $\eta = 0.5$.

The nanoscale effect is considered separately for different elasticity coefficients $\Lambda = 10, \Lambda = 100, \Lambda = 250, \Lambda = 500$.

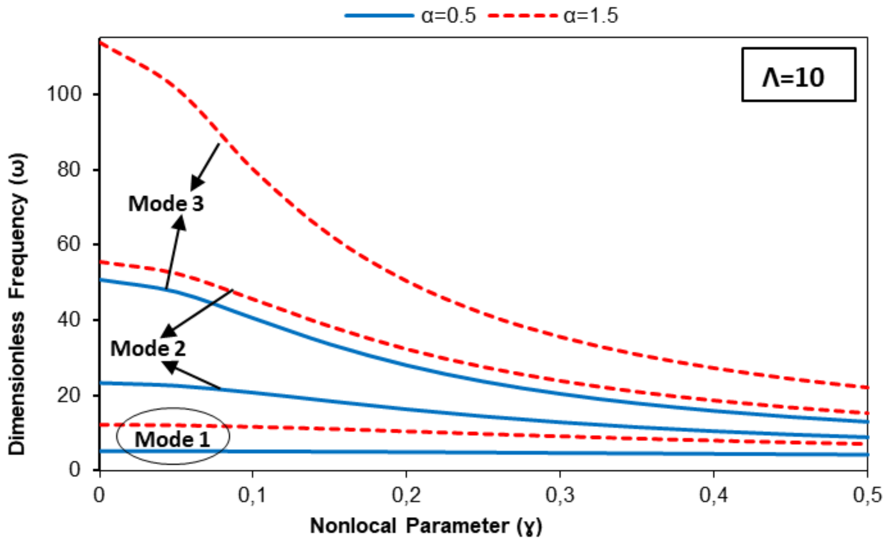

Figure 1. First three dimensionless frequencies of stepped nanobeam with various step ratios for versus nonlocal parameter

When Table 1 is examined, the elasticity values of the foundation increase, the natural frequency values also increase. This result shows parallelism with this study[10]. This parallelism strengthens the accuracy of the results of the study.

In Figure 1, the first three mode natural frequency values of the stepped nanobeam embedded in the elastic foundation are given for different non-local parameter values. Values are given in the graph for both $\alpha = 0.5$ and $\alpha = 1.5$. In addition, the step location has been determined as $\eta = 0.5$. In the results, it was seen that the non-local parameter had a negative effect on the natural frequency. So nonlocal parameter values increase, the natural frequency values decrease as seen in Table 1.

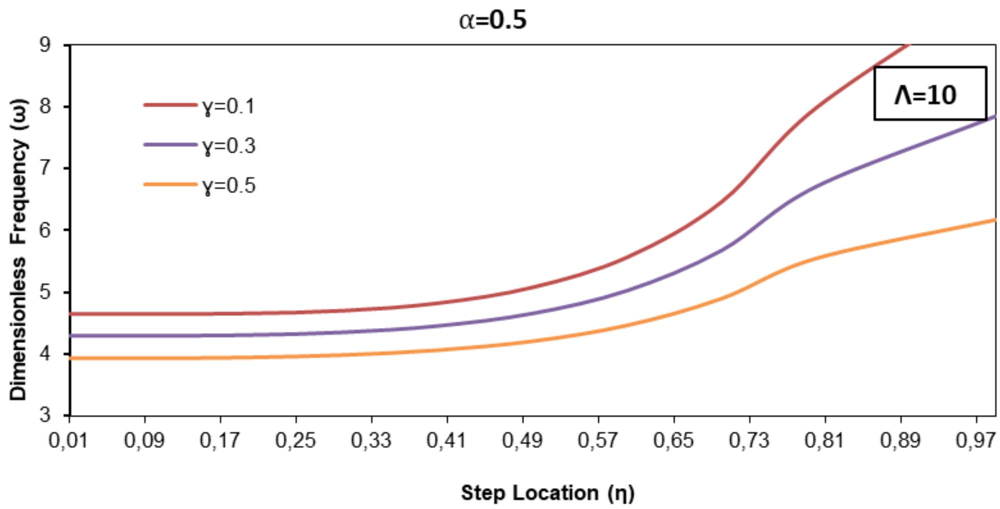

Figure 2. First dimensionless frequencies of stepped nanobeam with various the nonlocal parameter for versus step locations

In Figure 2, the variation of the fundamental frequencies of the stepped beam embedded in the elastic foundation with respect to the step position for different non-local parameter values $\gamma = 0.1 - 0.3 - 0.5$ are plotted. The following conclusions can be drawn from the plotted graph.

- It is seen that the natural frequency values increase as the step of the nanobeam movesfrom the starting point to the other end. It is understood that this situation is related to the increase of the thinner part of the stepped nanobeam (The step ratio $\alpha < 1$). This gives the same result for all non-local parameter values.

The present study investigates the free vibrations of stepped nanobeam embedded in elastic foundation. The results are presented in graphs and tables. It is seen that the natural frequencies of the first three modes decrease with the increase of the non-local parameter representing the effect of the nanoscale. The importance of the steps, which are thought to exist in the nature of the nano beam and indicate the originality of the study, was sought with its location and the ratio of the step. The results showed that the presence of the cascade contributes significantly to the natural frequency. In addition, the effects of the elastic foundation on the natural frequency were also observed. It is seen the natural frequency increase as the elastic foundation coefficient value is increased. There is no study of stepped nanobeam embedded in elastic foundation in the literature. Since this study is the first, it is expected that it will shed light on its field.

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