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Some Properties of Hessenberg Matrices with Conditional Elements

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Abstract – Hessenberg matrices are a class of structured matrices that play a fundamental role in various areas of mathematics and scientific computing. These matrices have diverse applications in numerical analysis, optimization, graph theory, and quantum mechanics, among others. This paper provides the determinants of the conditional Hessenberg matrices whose elements are the generalized bi-periodic Fibonacci numbers. Moreover, we find the inverse of some special type matrices which help us to find the determinant of the conditional Hessenberg matrices.

Keywords – Hessenberg Matrix, Determinant, Inverse, Triangular Matrix, Generalized Bi-Periodic Fibonacci Sequence.

I. Introduction

Matrices are used to represent linear transformations, such as rotations, scalings, and projections, and they find applications in diverse fields such as computer graphics, physics, economics, and engineering. Matrix inverses and matrix determinants are fundamental concepts in linear algebra that play a crucial role in various mathematical and practical applications. These concepts provide powerful tools for solving systems of linear equations, analysing the properties of matrices, and understanding transformations in vector spaces. Over the years, numerous researchers have explored and studied the intricate properties and calculations associated with matrix inverses and determinants.

In recent years, several researchers have investigated determinants and inverses of some Toeplitz matrices whose elements are special number sequences [1-4]. For example, Shen *et al.* obtained determinants and inverses of circulant matrices with Fibonacci and Lucas numbers [1]. The authors expressed the determinants of two types of circulant matrices by utilizing only Fibonacci and Lucas numbers. Stanimirović *et al.* investigated a generalized matrix of type s whose elements are defined by the generalized Fibonacci matrix [2]. Shen *et al.* defined a class of lower triangular Toeplitz matrices whose non-zero entries are the classical Horadam numbers [3]. The authors derived a convolution formula containing the Horadam numbers and by means of this formula, they obtained several identities. Shen *et al.* considered the lower Hessenberg matrix whose nonzero elements are the Horadam numbers. The authors computed the determinants and inverses of these matrices [4].

In recent years, several researchers have studied the applications and generalizations of the Fibonacci and Lucas numbers [5-8]. For example, Edson and Yayenie presented a notable generalization of the Fibonacci numbers which is called as bi-periodic Fibonacci sequence. Moreover, they derived an extended Binet formula, generating function and lots of identities of this sequence [6]. As a generalization of the bi-periodic Fibonacci numbers, Tan and Leung defined a generalization of Horadam sequence and they investigated some congruence properties of the generalized Horadam sequence [7].

II. Materials and Method

The relationship between number sequences and Hessenberg matrices unveils a connection between two seemingly distinct areas of mathematics. Number sequences, such as the Fibonacci or Lucas sequences, hold intrinsic patterns and properties that have intrigued mathematicians for centuries.

Before giving some relations between number sequences and Hessenberg matrices, we give some basic concepts.

Definition 2.1. [2] For $n \geq 2$, the second order recurrent sequence is defined by

$$U_n^{(a,b)} = AU_{n-1}^{(a,b)} + BU_{n-2}^{(a,b)},$$

where $A^2 + 4B > 0$ and the initial conditions are $U_0^{(a,b)} = a$ and $U_1^{(a,b)} = b$.

Definition 2.2. [7] For any nonzero real numbers a and b and for $n \geq 2$, the generalized bi-periodic Fibonacci numbers are defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2} & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2} & \text{if } n \text{ is odd} \end{cases},$$

with the arbitrary initial conditionals q_0 and q_1.

Definition 2.3. [2] The generalized Fibonacci matrix $\mathbb{U}_n^{(a,b,s)} = u_{i,j}^{(a,b,s)}$ is defined by

$$u_{i,j}^{(a,b,s)} = \begin{cases} U_{i-j+1}^{(a,b)}, & i-j+s \geq 0 \\ 0, & i-j+s < 0 \end{cases},$$

where $U_n^{(a,b)}$ is the second order recurrent sequence.

Definition 2.4. [4] The matrix $\mathcal{A} = [a_{i,j}] \in M_n$ is called as a lower Hessenberg matrix if $a_{i,j} = 0$ for $j > i+1$.

Theorem 2. 1.[7] For $n > 0$, the Binet formula of the generalized Fibonacci numbers is defined by

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{A\alpha^n - B\beta^n}{\alpha - \beta} \right),$$

where $A = \frac{q_1 - \frac{\beta}{a} q_0}{\alpha - \beta}$ and $B = \frac{q_1 - \frac{\alpha}{a} q_0}{\alpha - \beta}$.

Here we note that $\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}$ and $\beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}$ are the characteristic roots of the polynomial $x^2 - abx - ab$ and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function. Throughout the paper we assume that $\Delta = a^2b^2 + 4ab \neq 0$. Moreover, we have $\alpha + \beta = ab$ and $\alpha - \beta = \sqrt{a^2b^2 + 4ab}$ and $\alpha\beta = -ab$.

Definition 2.5. The conditional Hessenberg matrix $\mathbb{H}_n = h_{i,j}$ is defined by

$$h_{i,j} = \begin{cases} \left(\frac{b}{a} \right)^{\frac{\xi(i-j)}{2}} q_{i-j+1}, & i-j+1 \geq 0 \\ 0, & i-j+1 < 0 \end{cases},$$

where q_n is the generalized bi-periodic Fibonacci sequence.

III. Results

In this section, we give some properties of conditional Hessenberg matrices.

Theorem 3. 1. For the lower Hessenberg matrix $\mathbb{H}_n$, we have

$$\det[\mathbb{H}_n] = \frac{(q_1 - bq_0)^{n-2}(aq_1^2 - abq_0q_1 - bq_0^2)}{a}.$$

Proof. First, we investigate the case $q_1 = 0$. Let $\mathbb{A}_n = [a_{i,j}]$ and $\mathbb{B}_n = [b_{i,j}]$ be two $n \times n$ matrices which are defined as follows:

$$a_{i,j} = \begin{cases} 1, & i = j, \\ -\sqrt{ab}, & i = j+1, j > 1, \\ -1, & i = j+2, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$b_{i,j} = \begin{cases} 1, & i+j = 3, j \in \{1,2\} \\ & \text{or } i = j \geq 3, \\ 0, & \text{otherwise.} \end{cases}.$$

Thus, we can verify $\mathbb{A}_n \mathbb{H}_n \mathbb{B}_n = \mathbb{F}_n$, where

$$\mathbb{F}_n=\begin{pmatrix} \frac{\sqrt{b}q_0}{\sqrt{a}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\sqrt{b}q_0}{\sqrt{a}} & \frac{\sqrt{b}q_0}{\sqrt{a}} & 0 & \cdots & 0 \\ 0 & 0 & -bq_0 & \frac{\sqrt{b}q_0}{\sqrt{a}} & \cdots & 0 \\ 0 & 0 & 0 & -bq_0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{b}q_0}{\sqrt{a}} \\ 0 & 0 & 0 & 0 & \cdots & -bq_0 \end{pmatrix}$$

From the definition of the matrices $\mathbb{A}_n, \mathbb{B}_n$ and $\mathbb{F}_n$, we know that $\det \mathbb{A}_n = 1, \det \mathbb{B}_n = -1$ and $\det \mathbb{F}_n = \frac{(-1)^{n-2}\, q_0^n\, b^{n-1}}{a}$. Therefore, we get $\det[\mathbb{H}_n] = \frac{(-1)^{n-1}\, q_0^n\, b^{n-1}}{a}$ which satisfies the theorem.

Next, we investigate the case $q_1 \neq 0$. So, we define an $n \times n$ matrix $\mathbb{M}_\mathrm{n} = \left[m_{ij}\right]$ be of the form

$$m_{i,j}^{(a,b)} = \begin{cases} 1, & i=j, \\ -\frac{\sqrt{b}(q_0+aq_1)}{\sqrt{a}q_1}, & i=2, j=1 \\ -\sqrt{ab}, & i=j+1, j>1 \\ -1, & i=j+2 \\ 0, & \text{otherwise.} \end{cases}.$$

Hence, we have $\mathbb{M}_n\mathbb{H}_n = \mathbb{T}_n$, where

$$\mathbb{T}_n=\begin{pmatrix} q_1 & \frac{\sqrt{b}q_0}{\sqrt{a}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-bq_0^2-abq_0q_1+aq_1^2}{aq_1} & \frac{\sqrt{b}q_0}{\sqrt{a}} & 0 & \cdots & 0 \\ 0 & 0 & -bq_0+q_1 & \frac{\sqrt{b}q_0}{\sqrt{a}} & \cdots & 0 \\ 0 & 0 & 0 & -bq_0+q_1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{b}q_0}{\sqrt{a}} \\ 0 & 0 & 0 & 0 & \cdots & -bq_0+q_1 \end{pmatrix}$$

As $\det \mathbb{M}_\mathrm{n} = 1$ and $\det \mathbb{T}_n = \frac{(-bq_0+q_1)^{n-2}(-bq_0^2-abq_0q_1+aq_1^2)}{a}$, so we obtain

$$\det[\mathbb{H}_n] = \frac{(-bq_0+q_1)^{n-2}(-bq_0^2-abq_0q_1+aq_1^2)}{a}.$$

Therefore, the proof is completed.

Lemma 3. 1. Let $\mathbb{T}_n = [t_{ij}]$ be an $n \times n$ matrix which is defined in the Theorem 3.1.

If $-bq_0+q_1 \neq 0$ and $bq_0^2+abq_0q_1-aq_1^2 \neq 0$, then the inverse $\mathbb{T}_n^{-1} = [t'_{ij}]$ of the matrix $\mathbb{T}_n$ is equal to

$$t'_{ij} = \begin{cases} \frac{1}{q_1}, & i=j=1 \\ -\left(\frac{b}{a}\right)^{\frac{j-i}{2}-1} \frac{b\,(-q_0)^{j-i}{q_1}^{i-1}}{(-bq_0+q_1)^{j-2}(bq_0^2+abq_0q_1-aq_1^2)}, & 1 \le i \le 2, j>1 \\ \left(\frac{b}{a}\right)^{\frac{j-i}{2}} \frac{(-q_0)^{j-i}}{(-bq_0+q_1)^{j-i+1}}, & j \ge i > 2 \\ 0, & \text{otherwise} \end{cases}$$

Proof. Let $x_{i,j} = \sum_{l=1}^{n} \mathrm{t}_{i,l}\mathrm{t}'_{l,j}$. It is clear from Lemma 3.1 that $x_{1,1} = x_{2,2} = 1$, and for $i>j$ $x_{i,j}=0$. For the case $i=j>2$, we have

$$x_{i,i} = t_{i,i}t'_{i,i} = \left(-bq_0+q_1\right)\cdot\left(\frac{1}{-bq_0+q_1}\right) = 1.$$

In the case $j>1$, we obtain

$$\begin{aligned} x_{1,j} &= t_{1,1}t'_{1,j} + t_{1,2}t'_{2,j} \\ &= -q_1\cdot\left(\frac{b}{a}\right)^{\frac{j-3}{2}} \frac{b\,(-q_0)^{j-1}}{(-bq_0+q_1)^{j-2}(bq_0^2+abq_0q_1-aq_1^2)} \\ &\quad -\frac{\sqrt{b}q_0}{\sqrt{a}}\cdot\left(\frac{b}{a}\right)^{\frac{j-4}{2}} \frac{b\,(-q_0)^{j-2}\mathrm{q}_1}{(-bq_0+q_1)^{j-2}(bq_0^2+abq_0q_1-aq_1^2)} \\ &= 0. \end{aligned}$$

For $j>2$, we get

$$\begin{aligned} x_{2,j} &= \sum_{l=1}^{n} t_{2,l}t'_{l,j} = t_{2,2}t'_{2,j} + t_{2,3}t'_{3,j} \\ &= -\frac{(-bq_0^2-abq_0q_1+aq_1^2)}{aq_1}\cdot\left(\frac{b}{a}\right)^{\frac{j-4}{2}} \frac{b\,(-q_0)^{j-2}q_1}{(-bq_0+q_1)^{j-2}(bq_0^2+abq_0q_1-aq_1^2)} \\ &\quad +\frac{\sqrt{b}q_0}{\sqrt{a}}\cdot\left(\frac{b}{a}\right)^{\frac{j-3}{2}} \frac{(-q_0)^{j-3}}{(-bq_0+q_1)^{j-2}}. \\ &= 0. \end{aligned}$$

For the last case $n \ge j > i > 2$, we have

$$\begin{aligned} x_{i,j} &= t_{i,i}t'_{i,j} + t_{i,i+1}t'_{i+1,j} \\ &= (-bq_0+q_1)\cdot\left(\frac{b}{a}\right)^{\frac{j-i}{2}} \frac{(-q_0)^{j-i}}{(-bq_0+q_1)^{j-i+1}} \\ &\quad +\frac{\sqrt{b}q_0}{\sqrt{a}}\cdot\left(\frac{b}{a}\right)^{\frac{j-i-1}{2}} \frac{(-q_0)^{j-i-1}}{(-bq_0+q_1)^{j-i}} \\ &= 0. \end{aligned}$$

Therefore, we can obtain $\mathbb{T}_n\mathbb{T}_n^{-1} = \mathbb{I}_n$, where $\mathbb{I}_n$ is an $n \times n$ identity matrix. Similarly, we can verify $\mathbb{T}_n^{-1}\mathbb{T}_n = \mathbb{I}_n$. Hence, the proof is completed.

IV. CONCLUSION

In this paper, we investigate the determinants of lower triangular conditional Hessenberg matrices whose elements are the generalized bi-periodic Fibonacci numbers. We also computed the inverse of the matrix, which helps us find the determinant of the conditional Hessenberg matrices. This paper provides a brief overview of the properties and applications of conditional Hessenberg matrices, highlighting their significance in mathematical research and computational methods.

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