

Some Remarks on Rough I^K -Statistical Convergence of Sequences in Normed Linear Spaces

Ömer Kişi^{1,*} and Mustafa Yıldız²

^{1,*}Department of Mathematics, Faculty of Sciences, Bartın University, Turkey
ORCID ID 0000-0001-6844-3092

²Department of Mathematics, Faculty of Sciences, Bartın University, Turkey
ORCID ID 0000-0003-3367-7176
*(okisi@bartin.edu.tr)

Abstract –This study introduces the concept of rough I^* -statistical convergence in a normed linear space, extending the notion of rough I -statistical convergence. Furthermore, we propose the concept of rough I^K -statistical convergence in a more comprehensive framework. We examine the properties related to these novel concepts and explore the interconnections among rough I -statistical convergence, rough I^* -statistical convergence, and rough I^K -statistical convergence. By doing so, we enhance our understanding of these convergence modes and pave the way for their application in various mathematical contexts.

Keywords – Rough I -Statistical Convergence, Rough I^* -Statistical Convergence, Rough I^K -Statistical Convergence

I. INTRODUCTION

In the realm of real number sequences, the traditional notion of convergence has been extended to statistical convergence by Fast [1]. Since then, several advancements have emerged in this area from various researchers. One notable extension of statistical convergence is the introduction of I -convergence by Kostyrko et al. [2]. These developments have expanded our understanding of convergence in a statistical context and have paved the way for further investigations in this field.

A closely related convergence concept, referred to as I^* -convergence, was introduced by Kostyrko et al. [3]. The equivalence between this concept and the previously mentioned I -convergence has been established in [2], specifically when the ideal satisfies property (AP) . This result highlights the interplay between these two convergence notions and sheds light on the conditions under which they coincide.

Rough convergence was originally presented by Phu [4] in a finite-dimensional space. Building upon this notion, Dündar et al. [5] expanded the

framework in 2014 by introducing the concept of rough I -convergence. This novel concept combines the ideas of I -convergence and rough convergence. The concept of I^K -convergence in a topological space, considering two arbitrary ideals I and K on a set S , was introduced as a generalization of I^* -convergence by Mačaj and M. Sleziaak in [6]. In their study, they modified the condition (AP) and demonstrated that when the condition $AP(I, K)$ is satisfied, I -convergence implies I^K -convergence. Notably, they employed functions instead of sequences in their analysis, as utilizing functions can often lead to simplified notations. Banerjee and Paul [7] have investigated the concepts of rough I^* -convergence and rough I^K -convergence into the literature.

Within the framework of a normed linear space (NLS), we have introduced two new concepts: rough I^* -statistical convergence and rough I^K -statistical convergence. The latter, rough I^K -statistical convergence, can be viewed as a broader concept that encompasses rough I^* -statistical convergence. By introducing and exploring these

novel convergence modes, we contribute to the understanding and application of statistical convergence in the context of NLS.

Rough I^* -statistical convergence and rough I^K -statistical convergence in the context of NLSs carries substantial importance. These concepts expand upon the existing notion of rough I -statistical convergence, allowing for a more comprehensive understanding of convergence modes in mathematical analysis. By examining the properties associated with these newly introduced concepts, we deepen our comprehension of their

II. MAIN RESULTS

Throughout our discourse, we will consistently refer to an NLS denoted as $(X, \|\cdot\|)$, or simply X , over the field \mathbb{C} or \mathbb{R} . The ideals I, K are assumed to be non-trivial and admissible ideals on \mathbb{N} . Moreover, unless explicitly mentioned otherwise, the symbol r represents a non-negative real number.

Definition 2.1. We say that the sequence $\{y_n\}_{n \in \mathbb{N}}$ in $(X, \|\cdot\|)$ is rough I^* -statistical convergent of roughness degree ($r.deg.$) r to y if there is a set $M = \{p_i: p_i < p_{i+1}\}$ in the collection $F(I)$ of admissible ideals so that the subsequence $\{y_{p_i}\}_{i \in \mathbb{N}}$ is rough statistical convergent of $r.deg.$ r to y . In other words, for each $\rho > 0$, there exists a $i \in \mathbb{N}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\{k \leq t: \|y_{p_i} - y\| \geq r + \rho\}| = 0.$$

This is illustrated by $y_n \xrightarrow{r-I^*-st} y$.

The term "rough I^* -statistical limit" is used to refer to the limit of the sequence $\{y_n\}_{n \in \mathbb{N}}$ in terms of $r.deg.$ r . When r equals 0, it corresponds to the definition of I^* -statistical convergence of sequences in NLSs. It is important to note that the rough I^* -statistical limit of a sequence in NLSs is not unique. As a result, we define the rough I^* -statistical limit set of the sequence $\{y_n\}_{n \in \mathbb{N}}$ as follows:

$$I^* - st - LIM^r y_n = \left\{ y \in X: y_n \xrightarrow{r-st-I^*} y \right\}.$$

behavior and characteristics. Additionally, exploring the interconnections among rough I -statistical convergence, rough I^* -statistical convergence, and rough I^K -statistical convergence provides valuable insights into the relationships between these convergence modes. The outcomes of this study contribute to the advancement of mathematical knowledge and lay the foundation for their practical application in various mathematical contexts.

Definition 2.2. In an NLS $(X, \|\cdot\|)$, a sequence $\{y_n\}_{n \in \mathbb{N}}$ is called to be rough I^K -statistical convergent of $r.deg.$ r to y if there exists a set $M = \{p_i: p_i < p_{i+1}\}$ in the family of sets $F(I)$, so that the subsequence $\{y_{p_i}\}_{i \in \mathbb{N}}$ is rough $K | M$ -statistical convergent of $r.deg.$ r to y . Here, $K | M$ represents the trace of the ideal K on M , defined as $K | M = \{A \cap M: A \in K\}$. In other words, for all $\rho, \gamma > 0$, there is a $i \in \mathbb{N}$ such that

$$\left\{ t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_{p_i} - y\| \geq r + \rho\}| \geq \gamma \right\} \in K | M.$$

We indicate this by $y_n \xrightarrow{r-st-I^K} y$.

The term "rough I^K -statistical limit" is used to refer to the limit of the sequence $\{y_n\}_{n \in \mathbb{N}}$ in an NLS, considering the $r.deg.$ r and the ideal K . When r equals 0, it corresponds to the concept of I^K -statistical convergence of sequences in NLSs. It is important to note that for any $M \in F(I)$, the trace $K | M = \{A \cap M: A \in K\}$ of the ideal K on M also forms an ideal on \mathbb{N} . The rough I^K -statistical limit of a sequence in NLSs is not unique. So, we establish the rough I^K -statistical limit set of the sequence $\{y_n\}_{n \in \mathbb{N}}$ as

$$I^K - st - LIM^r y_n = \left\{ y \in X: y_n \xrightarrow{r-st-I^K} y \right\}.$$

If the ideal K consists of all finite subsets of \mathbb{N} , then Definition 2.1 and Definition 2.2 coincide. It is worth noting that if y is a rough I^* -statistical limit of a sequence $\{y_n\}_{n \in \mathbb{N}}$, then y is also a rough I^K -statistical limit of $\{y_n\}_{n \in \mathbb{N}}$. However, there are cases where y is a rough I^K -statistical limit of a

sequence $\{y_n\}_{n \in \mathbb{N}}$ in an NLS without being a rough I^* -statistical limit of the sequence $\{y_n\}_{n \in \mathbb{N}}$. This can be observed in the following example. Therefore, in general, for a sequence $\{y_n\}_{n \in \mathbb{N}}$ in an NLS and for any $r \geq 0$, we have $I^* - st - LIM^r y_n \subset I^K - st - LIM^r y_n$.

Example 2.1. Suppose we have a set \mathbb{N} decomposed into $\mathbb{N} = A \cup \bigcup_{i=1}^{\infty} A_i$, where $A = \{1, 3, 5, \dots\}$ and $A_i = \{2^k(2i-1) : k \in \mathbb{N}\}$. We observe that each of A_i 's are disjoint from each other and each of A_i 's are disjoint from A . Now, let's introduce I as the set of all subsets of \mathbb{N} that can intersect A and a finite number of A_i 's. Moving forward, we examine an alternative decomposition of \mathbb{N} as $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, where $D_j = \{2^{j-1}(2u-1) : u = 1, 2, \dots\}$. Each of D_j is infinite and it is obvious that $D_j \cap D_q = \emptyset$ for $j \neq q$. Assume that K be the ideal of all subsets of \mathbb{N} that intersect with only a finite number of D_j 's. So, K are non-trivial admissible ideal on \mathbb{N} . Now, let's examine a sequence in the real number space with the usual norm. We define this sequence as $y_n = \frac{1}{j}$ if $n \in D_j$. Take $M = \mathbb{N} \in F(I)$. Then $K \mid M = K$. Let's assume that we have an arbitrary positive number r . By applying the Archimedean property, for any arbitrary $\rho, \gamma > 0$, there is an $v \in \mathbb{N}$ such that $\rho > \frac{1}{v}$. Hence

$\left\{ t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : |y_i - (-r)| = |y_i + r| \geq r + \rho\}| \geq \gamma \right\} \subset D_1 \cup D_2 \cup \dots \cup D_v \in K = K \mid M$. As a result $-r \in I^K - st - LIM^r y_n$.

Let's explore the possibility of $-r \in I^* - st - LIM^r y_n$. In this case, there is a set $M = \{p_i : p_i < p_{i+1}\} \in F(I)$ for which the subsequence $\{y_n\}_{n \in M}$ of the sequence $\{y_n\}_{n \in \mathbb{N}}$ is rough statistical convergent to y of $r.deg. r$. Since $M \in F(I)$, we have $\mathbb{N} \setminus M = H \in I$. This implies that there is a positive integer t such that $H \subset A \cup A_1 \cup A_2 \cup \dots \cup A_t$, and therefore $A_i \subset M$ for each $i \geq t+1$.

Now, since each set A_i contains an element from each set D_i 's for $i \geq 2$, there is a $u > 0$ such that $y_{p_i} = \frac{1}{u}$ for infinitely many i when $p_i \in D_u$.

Considering $-r \in I^* - st - LIM^r y_n$, we can choose $\rho = \frac{1}{u+1}$. By the definition of rough statistical convergence, for each $\rho > 0$, there exists a $i \in \mathbb{N}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\{i \leq t : \|y_{p_i} - y\| \geq r + \rho\}| = 0. \rightarrow (*)$$

However, since $y_{p_i} = \frac{1}{u}$ for infinitely many k , the condition in (*) cannot hold. This leads to a contradiction. Hence, we conclude that $-r \notin I^* - st - LIM^r y_n$.

Theorem 2.1. If $\{y_n\}_{n \in \mathbb{N}}$ is rough I^* -statistical convergent with $r.deg. r$ to y , then it is also rough I -statistical convergent with $r.deg. r$ to y .

Proof. Suppose $\{y_n\}_{n \in \mathbb{N}}$ is rough I^* -statistical convergent with $r.deg. r$ to y . This implies the existence of a set $M = \{p_i : p_i < p_{i+1}\}$ such that $\{y_{p_i}\}_{i \in \mathbb{N}}$ is rough statistical convergent of $r.deg. r$ to y . For all $\rho > 0$, there exists a $i \in \mathbb{N}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\{i \leq t : \|y_{p_i} - y\| \geq r + \rho\}| = 0.$$

So, for any $\rho > 0$ there exists $m \in \mathbb{N}$ so that $\|y_{p_i} - y\| < r + \rho$ for all $i \geq m$. Then

$$\left\{ t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - y\| \geq r + \rho\}| \geq \gamma \right\} \subset \mathbb{N} \setminus M \cup \{p_1, p_2, \dots, p_{m-1}\} \rightarrow (i).$$

Since

$$\mathbb{N} \setminus M \cup \{p_1, p_2, \dots, p_{m-1}\} \in I,$$

it follows that

$$\left\{ t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - y\| \geq r + \rho\}| \geq \gamma \right\} \in I.$$

Therefore the sequence $\{y_n\}_{n \in \mathbb{N}}$ is rough I -statistical convergent of $r.deg. r$ to y .

According to Theorem 2.1, the rough I^* -statistical limit set with $r.deg. r$ is a subset of the rough I -

statistical limit set with the same $r.deg.$ r . However, the converse of Theorem 2.1 does not necessarily hold true. In other words, if a sequence $\{y_n\}_{n \in \mathbb{N}}$ is rough I -statistical convergent with some $r.deg.$ r to y , it may not be rough I^* -statistical convergent with the same $r.deg.$ r to y . This fact becomes apparent from the following example.

Example 2.2. Let's consider a decomposition of the set of \mathbb{N} as $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, where $D_j = \{2^{(j-1)}(2u - 1) : u = 1, 2, \dots\}$. Each set D_j is infinite and disjoint from each other. Now, let I be the class of subsets of \mathbb{N} that intersect with only a finite number of D_j 's. It can be shown that I is an admissible ideal on \mathbb{N} .

We present a sequence $\{y_n\}_{n \in \mathbb{N}}$ in the real number space with the usual norm, such that $y_n = \frac{1}{j}$ if n belongs to D_j . Let $r \geq 0$. For any arbitrarily chosen $\rho > 0$, we can find $v > 0$ such that $\rho > \frac{1}{v}$. Consequently, we obtain $[-r, r] \subset I - st - LIM^r y_n$, since the set $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - y\| \geq r + \rho\}| \geq \gamma\right\}$ is contained in $D_1 \cup D_2 \cup \dots \cup D_v$, which belongs to I for any y in the interval $[-r, r]$.

Now, let's assume that the aforementioned sequence is rough I^* -statistically convergent to $-r$, where $-r$ has the same $r.deg.$ r . This implies the existence of a set $M = \{p_i : p_i < p_{i+1}\}$ such that $\{y_{p_i}\}_{i \in \mathbb{N}}$ is rough statistical convergent to $-r$ of $r.deg.$ r . Since $M \in F(I)$, we have $\mathbb{N} \setminus M = H \in I$. Therefore, there is a $t > 0$ so that H is a subset of $D_1 \cup D_2 \cup \dots \cup D_t$, and thus $D_{t+1} \subset M$. Consequently, for any $p_i \in D_{t+1}$, we get $y_{p_i} = \frac{1}{(t+1)^{t+1}}$. Now, selecting $\rho = \frac{1}{(t+2)^{t+1}}$, we can observe that for $p_i \in D_{t+1}$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\{i \leq t : \|y_{p_i} - y\| \geq r + \rho\}| = 0.$$

for infinitely many values of i . Therefore, the sequence $\{y_n\}_{n \in \mathbb{N}}$ is not rough I^* -statistical convergent of $r.deg.$ r to $-r$, even though $-r$ belongs $I - st - LIM^r y_n$.

For a sequence $\{y_n\}_{n \in \mathbb{N}}$ in an NLS, the rough I -statistical limit of the sequence with $r.deg.$ r is also a rough I^* -statistical limit with the same $r.deg.$ r ,

provided that the ideal I supplies condition (AP). To establish this result, we will make use of the following lemma.

Lemma 2.1. Suppose that $\{A_n\}_{n \in \mathbb{N}}$ be a countable family of subsets of \mathbb{N} , where all A_n belongs to $F(I)$, the filter associated with an admissible ideal I that satisfies property (AP). Then, there exists a set $B \subset \mathbb{N}$ such that $B \in F(I)$ and the set $B \setminus A_n$ is finite for all $n \in \mathbb{N}$.

Theorem 2.2. Assume that I be an ideal that satisfies property (AP), and consider a sequence $\{y_i\}_{i \in \mathbb{N}} \in (X, \|\cdot\|)$. If y is a rough I -statistical limit of the sequence $\{y_i\}_{i \in \mathbb{N}}$ with a certain $r.deg.$ r , then y is also a rough I^* -statistical limit of the sequence $\{y_i\}_{i \in \mathbb{N}}$ with the same $r.deg.$ r .

Proof. Consider an ideal I , on the set of \mathbb{N} , satisfying the condition (AP). Let $\{y_i\}_{i \in \mathbb{N}} \in (X, \|\cdot\|)$. Suppose y is a rough I -statistical limit of the sequence $\{y_i\}_{i \in \mathbb{N}}$ with a $r.deg.$ of r . This implies that for any $\rho, \gamma > 0$, the set $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - y\| \geq r + \rho\}| \geq \gamma\right\}$ belongs to the ideal I .

Let $l \in \mathbb{R}^+$, and note that $l/s \in \mathbb{R}^+$ for each $s \in \mathbb{N}$. We define A_s as the set $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - y\| \geq r + \frac{l}{s}\}| < \gamma\right\}$ for every $s \in \mathbb{N}$. It follows that $A_s \in F(I)$ for all $s \in \mathbb{N}$. Additionally, according to lemma 2.1, there is a set $B \subset \mathbb{N}$ that belongs to the $F(I)$ class and $B \setminus A_s$ is finite for all $s \in \mathbb{N}$.

For any arbitrary $\rho > 0$, there exists a $j \in \mathbb{N}$ such that $\rho < \frac{l}{j}$. Since $B \setminus A_j$ is finite, we can find a $t = t(j) \in \mathbb{N}$ such that for all $i \in B$ with $i \geq t$, we have $i \in B \cap A_j$. Consequently, for all $i \in B$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\{i \leq t : \|y_i - y\| \geq r + \rho\}| = 0.$$

This implies that the subsequence $\{y_i\}_{i \in B}$ is rough statistical convergent of $r.deg.$ r to y . So, y is also a rough I^* -statistical limit of $r.deg.$ r . Hence, the result follows.

Corollary 2.1. Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in an NLS $(X, \|\cdot\|)$. Consider I , an ideal on \mathbb{N} , satisfying the condition (AP). We aim to prove that both the rough I -statistical limit set of $r.deg. r$ and the rough I^* -statistical limit set of $r.deg. r$ for the sequence $\{y_n\}_{n \in \mathbb{N}}$ are equal.

By Theorem 2.1 and Theorem 2.2, we can conclude that the result holds.

The rough I -statistical limit set of a sequence $\{y_n\}_{n \in \mathbb{N}}$ in an NLS is a subset of the rough I -statistical limit set of a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$. However, the rough I^* -statistical limit set of a sequence $\{y_n\}_{n \in \mathbb{N}}$ in an NLS may not necessarily be a subset of the rough I^* -statistical limit set of a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$.

Theorem 2.3. If we have an ideal I that supplies condition (AP), then the rough I^* -statistical limit set of a sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg. r$ is contained in the rough I^* -statistical limit set of a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ with the same $r.deg. r$.

Proof. Consider y to be a rough I^* -statistical limit of a sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg. r$. Since a rough I^* -statistical limit is also a rough I -statistical limit of $\{y_n\}_{n \in \mathbb{N}}$, it follows that y is a rough I -statistical limit of $\{y_n\}_{n \in \mathbb{N}}$. Furthermore, as the rough I -statistical limit of a sequence $\{y_n\}_{n \in \mathbb{N}}$ is a subset of the rough I -statistical limit of a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$, we can conclude that y is also a rough I -statistical limit of the subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$.

Now, since the ideal I supplies the condition (AP), y is also a rough I^* -statistical limit of the subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$. Therefore, we can affirm that x is both a rough I -statistical limit and a rough I^* -statistical limit of the subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$.

Theorem 2.4. Let's consider an NLS $(X, \|\cdot\|)$, and let I and K be two admissible ideals on the set of \mathbb{N} . Suppose we have a sequence $\{y_n\}_{n \in \mathbb{N}}$ in X that is rough I^K -statistical convergent to y , with a $r.deg. r$. We can state that if K is a subset of I , then the sequence $\{y_n\}_{n \in \mathbb{N}}$ is also rough I -statistical convergent to y , maintaining the same $r.deg. r$.

Proof. Consider two admissible ideals, I and K , on the set of \mathbb{N} , where K is a subset of I . Suppose we have a sequence $\{y_n\}_{n \in \mathbb{N}}$ that is rough I^K -statistical convergent to y with a $r.deg. r$. According to the theorem, there is a set $M = \{p_i: p_i < p_{i+1}\} \in F(I)$ so that for any $\rho, \gamma > 0$, the set $A(\rho, \gamma) = \{t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_{p_i} - y\| \geq r + \rho\}| \geq \gamma\} \in K \mid M$. It is also given that $A(\rho, \gamma) = \{t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_{p_i} - y\| \geq r + \rho\}| \geq \gamma\} = K_1 \cap M$ for some $K_1 \in K$. Since K is an ideal and $K_1 \cap M \in K$, we can conclude that $K_1 \cap M \in K$. Additionally, $\{t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_i - y\| \geq r + \rho\}| \geq \gamma\} \subset (K_1 \cap M) \cup \mathbb{N} \setminus M$. As $\mathbb{N} \setminus M \in I$ and $K \subset I$, it follows that $(K_1 \cap M) \cup \mathbb{N} \setminus M \in I$. So $\{t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_i - y\| \geq r + \rho\}| \geq \gamma\} \in I$. Thus, the result is established.

Conversely, the validity of the converse part of Theorem 2.4 holds true as well. If a rough I^K -statistical limit y of a sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg. r$ implies that y is also a rough I -statistical limit with the same $r.deg. r$, then $K \subset I$. To establish this, we require the utilization of the following lemma.

Lemma 2.2. Suppose I and K are ideals on the set of natural numbers, \mathbb{N} . If a sequence $\{y_n\}_{n \in \mathbb{N}}$ has a rough K -statistical limit, characterized by a $r.deg. r$, then it is also a rough I^K -statistical limit with the same $r.deg. r$.

Proof. Consider two ideals, I and K , on the set of \mathbb{N} , and let $r \geq 0$. Suppose y is a rough K -statistical limit of the sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg. r$, denoted by $y \in K - st - LIM^r y_n$. It means that for any $\rho, \gamma > 0$, the set $\{t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_i - y\| \geq r + \rho\}| \geq \gamma\}$ belongs to K . Now, since the empty set belongs to I , we can conclude that $\mathbb{N} \in F(I)$. Let $M = \{p_i: p_i < p_{i+1}\} = \mathbb{N} \in F(I)$. This implies that the subsequence $\{y_{p_i}\}$ is equal to the original sequence $\{y_n\}$ and $K \mid M = K$. Therefore, the set $\{t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_{p_i} - y\| \geq r + \rho\}| \geq \gamma\}$ is equivalent to $\{t \in \mathbb{N}: \frac{1}{t} |\{i \leq t: \|y_i - y\| \geq r + \rho\}| \geq \gamma\} \in K = K \mid M$. As a

result, we can say that $y \in I^K - st - LIM^r y_n$. Thus, the desired result is obtained.

Theorem 2.5. Suppose the rough I^K -statistical limit of a sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg.$ of r is y . If y is also a rough I -statistical limit of the same sequence $\{y_n\}_{n \in \mathbb{N}}$ with the same $r.deg.$ of r , then it implies that K is a subset of I .

Proof. Assume that K is not a subset of I , and r is a non-negative real number. This implies the existence of a set $A \in K \setminus I$. Let's select u and v from the NLS X , where $\|u\| = 1$ and $v = (r + 2)u$. We observe that $\|u - v\| \geq r + \rho$ for $0 < \rho \leq 1$ and $\|u - v\| < r + \rho$ for $\rho > 1$.

Next, we shall establish a sequence $\{y_n\}_{n \in \mathbb{N}}$ in the following manner:

$$y_n = \begin{cases} u + r, & n \in \mathbb{N} \setminus A \\ y, & n \in A \end{cases}.$$

For any $\rho > 0$, the set $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - u\| \geq r + \rho\}| \geq \gamma\right\}$ will either be equal to the set A (when $0 < \rho \leq 1$) or the empty set \emptyset (when $\rho > 1$). Since K is an admissible ideal and $A \in K$, it follows that $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - u\| \geq r + \rho\}| \geq \gamma\right\} \in K$. Thus, u is a rough K -statistical limit of the sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg.$ of r .

Now, using lemma 2.2, we can conclude that u is also a rough I^K -statistical limit of the sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg.$ of r . However, considering the case where $0 < \rho \leq 1$, we see that $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - u\| \geq r + \rho\}| \geq \gamma\right\} = A$, and since $A \notin I$, it follows that $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - u\| \geq r + \rho\}| \geq \gamma\right\} \notin I$. This implies that u is not a rough I -statistical limit with a $r.deg.$ of r .

We have reached a contradiction, as our assumption stated that u is a rough I -statistical limit of $\{y_n\}_{n \in \mathbb{N}}$. Therefore, we can conclude that $K \subset I$.

Corollary 2.2. The rough I^K -statistical limit set of a sequence $\{y_n\}_{n \in \mathbb{N}}$ with a $r.deg.$ of r is a subset of the I -statistical limit set if and only if K is a subset of I .

In general, the fact that y is a rough I -statistical limit of a sequence $\{y_n\}_{n \in \mathbb{N}}$ in a normed linear space (NLS) does not necessarily imply that y is also a rough I^K -statistical limit of the same sequence. To support this claim, we provide the following example.

Example 2.3. Consider the ideal I , as described in Example 2.2. Suppose that K be the ideal on \mathbb{N} , defined as the collection of all subsets of \mathbb{N} whose natural density is zero. We create a sequence $\{y_n\}$ in \mathbb{R} with the usual norm, where $y_n = \frac{1}{j}$ if $n \in D_j$. Let $r \geq 0$.

Now, let $\rho > 0$. We can find an $l \in \mathbb{N}$ such that $\rho > 1/l$. It is clear that $[-r, r]$ is contained in the I -statistical set of $\{y_n\}$ with a $r.deg.$ of r , indicated by $I - st - LIM^r y_n$, since $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_i - y\| \geq r + \rho\}| \geq \gamma\right\}$ is a subset of $D_1 \cup D_2 \cup \dots \cup D_l$, which belongs to I for any $y \in [-r, r]$.

Suppose, for contradiction, that $-r$ is a rough I^K -statistical limit of $\{y_n\}$ with a $r.deg.$ of r . This implies the existence of an $M = \{p_i : p_i < p_{i+1}\} \in F(I)$ such that the subsequence $\{y_{p_i}\}$ is rough $K \mid M$ -statistical convergent to $-r$ with a $r.deg.$ of r .

Since $N \setminus M = H$ belongs to I , there exists a $s \in \mathbb{N}$ such that H is a subset of $D_1 \cup D_2 \cup \dots \cup D_s$. Consequently, for all $i \geq s + 1$, we have $D_i \subset M$.

Now, consider $\rho = \frac{1}{s+1}$, and $\gamma > 0$. Then,

$$\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_{p_i} - y\| \geq r + \frac{1}{s+1}\}| \geq \gamma\right\} = \{i \in \mathbb{N} : p_i \in D_{s+1}\}.$$

As $D_{s+1} = \{2^s(2u - 1) : u = 1, 2, \dots\}$, and the natural density of D_{s+1} is $\frac{1}{2^{s+1}}$. So, $\delta\left(\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_{p_i} - y\| \geq r + \frac{1}{s+1}\}| \geq \gamma\right\}\right) \neq 0$. As a result, we obtain $\left\{t \in \mathbb{N} : \frac{1}{t} |\{i \leq t : \|y_{p_i} - y\| \geq r + \frac{1}{s+1}\}| \geq \gamma\right\} \notin K \mid M$, since natural density of all set belongs to $K \mid M$ is also zero. So $-r$ is not a rough I^K -statistical limit of $\{y_n\}$ of $r.deg.$ r .

Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence, and let I be an ideal satisfying condition (AP). If y is a rough I -

statistical limit of $\{y_n\}_{n \in \mathbb{N}}$, then y is also a rough I^K -statistical limit of $\{y_n\}_{n \in \mathbb{N}}$.

Theorem 2.6. Consider two admissible ideals, denoted as I and K , on the set of \mathbb{N} . Assume that I satisfies the condition (AP). Additionally, let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence in an NLS $(X, \|\cdot\|)$. Then $y \in I - st - LIM^r y_n$ implies $y \in I^K - st - LIM^r y_n$.

Proof. Let I and K be ideals on \mathbb{N} so that the ideal I supplies the condition (AP). Assume that $\{y_n\}_{n \in \mathbb{N}}$ be sequence so that $y \in I - st - LIM^r y_n$. since I holds the condition (AP), so $y \in I^* - st - LIM^r y_n$. Now as $I^* - st - LIM^r y_n \subset I^K - st - LIM^r y_n$, therefore $y \in I^K - st - LIM^r y_n$.

CONCLUSION

In this study, we have introduced and explored the concept of rough I^* -statistical convergence in a normed linear space, thereby extending the existing notion of rough I -statistical convergence. Additionally, we have proposed the concept of rough I^K -statistical convergence, which provides a more comprehensive framework for understanding these convergence modes.

Through our examination of the properties associated with these novel concepts, we have gained insights into their characteristics and established their interconnections. We have identified relationships between rough I -statistical convergence, rough I^* -statistical convergence, and rough I^K -statistical convergence, thus contributing to our understanding of these convergence modes.

By expanding our knowledge of these convergence modes, we have opened up avenues for their application in various mathematical contexts. The enhanced understanding and interconnections provided by our study offer valuable insights for future research in this field. These convergence modes have the potential to find applications in diverse mathematical areas, enabling further advancements in related theories and applications.

Overall, this study has expanded our understanding of rough I^* -statistical convergence, introduced the concept of rough I^K -statistical convergence, and elucidated the relationships between these convergence modes. This research paves the way for further investigations and applications of these

concepts, benefiting the broader mathematical community and promoting advancements in related fields.

REFERENCES

- [1] H. Fast, *Sur la convergence statistique*, Colloq. Math., 2 (1951), 241-244.
- [2] P. Kostyrko, T. Šalát, W. Wilczyński, *I-convergence*, Real Anal. Exchange, 26 (2) (2000/2001), 669-686.
- [3] P. Kostyrko, M. Macaj, and T. Šalát, *Statistical convergence and I-convergence*, 1999, [http://thales.doa.fmph.uniba.ak/macaj/ICON.pdf](http://thales.doa.fmph.uniba.sk/macaj/ICON.pdf).
- [4] H. X. Phu, *Rough convergence in normed linear space*, Numer. Funct. Anal. Optim., 22 (2001), 199-222.
- [5] E. Dündar and C. Çakan, *Rough I-convergence*, Demonstr. Math., 47 (3) (2014), 638-651.
- [6] M. Mačaj and M. Slezniak, *I^K -convergence*, Real Anal. Exchange, 36 (1) (2010/2011), 177- 194.
- [7] A. K. Banerjee and A. Paul, *On rough I^* and I^K -convergence of sequences in normed linear spaces*, Facta Univ., Math. Inform., 37 (3) (2022), 541-557.