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Conoid Surfaces Family in Three-Dimensional Euclidean Space

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Abstract – In this research, we investigate a specific family of conoid surfaces within the three-dimensional Euclidean space $\mathbb{E}^3$. We consider the differential geometry of the family. We determine the curvatures of these particular surfaces. Moreover, we provide the necessary conditions for minimality within this framework. Additionally, we compute the Laplace−Beltrami operator for this family and present an example.

Keywords – Euclidean 3-space, conoid surfaces family, Gauss map, curvatures, Laplace−Beltrami operator

I. Introduction

Chen [4, 5, 6, 7] initially introduced the concept of finite order sub-manifolds (*SM*) of immersed into Euclidean m-space $\mathbb{E}^m$ or pseudo-Euclidean m-space $\mathbb{E}_v^m$ by utilizing a finite set of eigenfunctions derived from their Laplacian. Since then, this topic has undergone extensive scrutiny and investigation.

Takahashi demonstrated that a Euclidean submanifold is classified as 1-type if and only if it is minimal or minimal within a hypersphere of $\mathbb{E}^m$. The minimal *SM* were provided by Lawson [20]. Garay subsequently [16] examined Takahashi's theorem in $\mathbb{E}^m$. Aminov [2] extensively explored the geometry of *SM*. Chen et al. [8], over the span of four decades, dedicated their research efforts to investigating 1-type *SM* and the 1-type Gauss map (*Gm*) within the realm of space forms.

In the three-dimensional Euclidean space, denoted as $\mathbb{E}^3$, Takahashi [22] conducted an exploration of minimal surfaces. Within this context, spheres and surfaces with minimal sections are the exclusive types of surfaces identified. Ferrandez et al. [14] determined that surfaces with specific characteristics are either minimal sections of a sphere or a right circular cylinder. Choi and Kim [10] focused their study on the minimal helicoid, which exhibits a pointwise 1-type *Gm* of the first kind. Garay [15] derived a category of finite type surfaces that are based on revolution. Dillen et al. [11] investigated a distinct set of surfaces characterized by certain properties, including minimal surfaces, spheres, and circular cylinders.

Additionally, the extensive research has been carried out by Berger and Gostiaux [3], Do Carmo [12], Gray [17], and Kreyszig [18] on the right conoids in three-dimensional space. These studies have focused on various right conoids, such as the helicoid, Whitney umbrella, Wallis's conical edge, Plücker's conoid, and hyperbolic paraboloid.

The objective of this study is to examine the characteristics of the conoid surfaces family within the three-dimensional Euclidean space $\mathbb{E}^3$. Our specific goals are to calculate the matrices corresponding to the fundamental form, *Gm*, and shape operator (*SO*) of this family. By utilizing the Cayley−Hamilton theorem, we aim to ascertain the curvatures of these surfaces. Furthermore, we strive to establish the criteria for determining minimality within this framework. Additionally, our aim is to explore the relationship between the Laplace−Beltrami operator and these types of surfaces.

In Section 2, a detailed explanation of the fundamental principles and concepts underlying three-dimensional Euclidean geometry is provided.

Section 3 is dedicated to the presentation of the curvature formulas applicable to surfaces in $\mathbb{E}^3$.

In Section 4, a comprehensive definition of conoid surfaces family is offered, emphasizing their distinctive properties and characteristics.

In Section 5, the focus shifts to the discussion of the Laplace−Beltrami operator for a smooth function in $\mathbb{E}^3$, and the application of the previously examined surfaces in its computation.

Finally, we conclude the research in the last section.

II. Preliminaries

In this paper, we use the following notations, formulas, Eqs., etc.

Let M be an oriented hypersurface in $\mathbb{E}^{n+1}$ with its *SO* S, position vector x. Consider a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ consisting of principal directions of M coinciding with the principal curvature k_i for $i = 1,2,\dots,n$.

Let the dual basis of this frame field be $\{\theta_1, \theta_2, \dots, \theta_n\}$. Then, the first structural Eq. of Cartan is determined by

$$d\theta_i = \sum_{i,j=1}^{n} \theta_j \wedge \omega_{ij},$$

where ω_{ij} indicates the connection forms coinciding with the chosen frame field. By the Codazzi Eq., we derive the Eqs.:

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j),$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l),$$

for different $i, j, l = 1,2,\dots,n$.

We let

$$s_j = \sigma_j(k_1, k_2, \dots, k_n),$$

where σ_j denotes the j-th elementary symmetric function defined by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} a_{i_1} a_{i_2} \dots a_{i_j}.$$

We consider the notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, \dots, k_n).$$

We have

$$r_i^0 = 1,$$

and

$$s_{n+1} = s_{n+2} = \dots = 0.$$

The function s_k is referred to as the k-th mean curvature (*MC*) of the oriented hypersurface M. The *MC* is described by

$$H = \frac{1}{n} s_1,$$

and the Gauss−Kronecker curvature of M is determined by

$$K = s_n.$$

If $s_j \equiv 0$, the hypersurface M is known as j-minimal.

In Euclidean $(n+1)$-space, to obtain the curvature formulas $\mathcal{K}_i$ (See [1] and [19] for details.), $i = 0,1,\dots,n$, we have the following characteristic polynomial Eq.:

$$P_S(\lambda) = 0,$$

that is,

$$\sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = \det(S - \lambda)\mathcal{I}_n = 0. \quad (2.1)$$

Here, $i = 0,1,\dots,n$, $\mathcal{I}_n$ denotes the identity matrix. Hence, we reveal the curvature formulas

$$\binom{n}{i}\mathcal{K}_i = s_i.$$

Consider the immersion $\mathfrak{I} = \mathfrak{I}(u,v)$ from $M^2 \subset \mathbb{E}^2$ to $\mathbb{E}^3$.

Definition 1. An inner product of two vectors

$$\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3) \text{ and } \boldsymbol{\varrho} = (\varrho^1, \varrho^2, \varrho^3)$$

of $\mathbb{E}^3$ is determined by

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varrho} \rangle = \sigma^1\varrho^1 + \sigma^2\varrho^2 + \sigma^3\varrho^3.$$

Definition 2. A vector product of

$$\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3) \text{ and } \boldsymbol{\varrho} = (\varrho^1, \varrho^2, \varrho^3)$$

of $\mathbb{E}^3$ is defined by

$$\boldsymbol{\sigma} \times \boldsymbol{\varrho} = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ \sigma^1 & \sigma^2 & \sigma^3 \\ \varrho^1 & \varrho^2 & \varrho^3 \end{pmatrix}.$$

Definition 3. The matrix

$$(\mathfrak{g}_{ij})^{-1}(\mathfrak{h}_{ij})$$

determines the *SO* matrix S of surface $\mathfrak{I}$ in Euclidean 3-space $\mathbb{E}^3$, where

$$(\mathfrak{g}_{ij})_{2\times 2} \text{ and } (\mathfrak{h}_{ij})_{2\times 2}$$

describe the first and the second fundamental form matrices, respectively, and

$$\mathfrak{g}_{ij} = \langle \mathfrak{I}_i, \mathfrak{I}_j \rangle,\ \ \mathfrak{h}_{ij} = \langle \mathfrak{I}_{ij}, \boldsymbol{\mathcal{G}} \rangle,\ \ i,j = 1,2,$$

$\mathfrak{I}_u = \frac{\partial \mathfrak{I}}{\partial u}$ when $i = 1$, $\mathfrak{I}_{uv} = \frac{\partial^2 \mathfrak{I}}{\partial u \partial v}$ when $i = 1$ and $j = 2$, etc., e_k denotes the natural base elements of $\mathbb{E}^3$, and

$$\boldsymbol{\mathcal{G}} = \frac{\mathfrak{I}_u \times \mathfrak{I}_v}{\|\mathfrak{I}_u \times \mathfrak{I}_v\|} \qquad (2.2)$$

determines the *Gm* of the surface $\mathfrak{I}$.

III. Curvatures in Three-Space

In this section, we obtain the curvature formulas of any surface $\mathfrak{I} = \mathfrak{I}(u,v)$ *in* $\mathbb{E}^3$.

Theorem 1. *A surface $\mathfrak{I}$ in $\mathbb{E}^3$ exhibits the following formulas,*

$$\mathcal{K}_0 = 1,\ \ 2\mathcal{K}_1 = -\frac{\mathfrak{p}_1}{\mathfrak{p}_2},\ \ \mathcal{K}_2 = \frac{\mathfrak{p}_0}{\mathfrak{p}_2} \qquad (3.1)$$

where

$$\mathfrak{p}_2\lambda^2 + \mathfrak{p}_1\lambda + \mathfrak{p}_0 = 0$$

describes the characteristic polynomial Eq. of the shape operator matrix,

$$\mathfrak{p}_2 = \det(\mathfrak{g}_{ij}),\ \ \mathfrak{p}_0 = \det(\mathfrak{h}_{ij}),$$

$(\mathfrak{g}_{ij})_{2\times 2}$ *represent the first fundamental form matrix and* $(\mathfrak{h}_{ij})_{2\times 2}$ *represent the second fundamental form matrix.*

Proof. The matrix

$$(\mathfrak{g}_{ij})^{-1}(\mathfrak{h}_{ij})$$

describes the shape operator matrix of surface $\mathfrak{I}$ in Euclidean 3-space $\mathbb{E}^3$. We reveal the characteristic polynomial Eq.

$$\det(S - \lambda \mathcal{I}_2) = 0.$$

Thus, we obtain the curvatures

$$\binom{2}{0}\mathcal{K}_0 = 1,$$

$$\binom{2}{1}\mathcal{K}_1 = k_1 + k_2 = -\frac{\mathfrak{p}_1}{\mathfrak{p}_2},$$

$$\binom{2}{2}\mathcal{K}_2 = k_1 k_2 = \frac{\mathfrak{p}_0}{\mathfrak{p}_2}.$$

Definition 4. *A surface $\mathfrak{I}$ is called j-minimal if* $\mathcal{K}_j = 0$, *where* $j = 1,2$.

Theorem 2. *A surface* $\mathfrak{I} = \mathfrak{I}(u,v)$ *in* $\mathbb{E}^3$ *has the following relation*

$$\mathcal{K}_0 \mathbb{IIII} - 2\mathcal{K}_1 \mathbb{III} + \mathcal{K}_2 \mathbb{I} = \mathcal{O}_2,$$

where $\mathbb{I}, \mathbb{III}, \mathbb{IIII}$ *determines the fundamental form matrices,* $\mathcal{O}_2$ *represents the zero matrix having order 2 of the surface.*

Proof. Regarding $n = 2$ in (2.1), it runs.

IV. Conoid Surfaces Family in $\mathbb{E}^3$

In this section, we define the conoid surfaces family (*CSF*), then find its differential geometric properties in Euclidean 3-space $\mathbb{E}^3$.

A ruled surface

$$\mathfrak{r}(u,v) = p(v) + u\, q(v)$$
$$= (0,0,\beta(v))$$
$$+u(\cos\alpha(v), \sin\alpha(v), 0)$$

is termed a *right conoid* in $\mathbb{E}^3$ if it can be generated by the translation of a straight line that intersects a fixed straight line, while ensuring that the lines maintain a perpendicular relationship throughout the generation process. By considering the *xy*-plane as the perpendicular plane and selecting the *z*-axis as the reference line, the parametric Eq. for the right conoid is given by

$$\mathfrak{r}(u,v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u\cos\alpha(v) \\ u\sin\alpha(v) \\ \beta(v) \end{pmatrix},$$

Helicoid, Whitney umbrella, Wallis's conical edge, Plücker's conoid, hyperbolic paraboloid are each examples of a right conoid surface. For details see Berger and Gostiaux [3], Do Carmo [12], Gray [17], Kreyszig [18].

Definition 5. *A CSF is an immersion* $\mathfrak{I}$ *from* $M^2 \subset \mathbb{E}^2$ to $\mathbb{E}^3$ *with the reference line z, defined by*

$$\mathfrak{I}(u,v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(u)\cos g(v) \\ f(u)\sin g(v) \\ h(v) \end{pmatrix} \quad (4.1)$$

where

$$f = f(u), \;\; g = g(v), \;\; h = h(v)$$

denote the differentiable functions.

Taking the first derivatives of *CSF* determined by Eq. (4.1) w.r.t. u, v, respectively, we obtain the first fundamental form matrix

$$(\mathfrak{g}_{ij}) = \begin{pmatrix} f_u^2 & 0 \\ 0 & f^2 g_v^2 + h_v^2 \end{pmatrix} \quad (4.2)$$

and

$$f_u^2 = \left(\frac{\partial f}{\partial u}\right)^2, \;\; g_v^2 = \left(\frac{\partial g}{\partial v}\right)^2, \;\; h_v^2 = \left(\frac{\partial h}{\partial v}\right)^2.$$

Hence,

$$\det(\mathfrak{g}_{ij}) = f_u^2 \mathcal{W},$$

where

$$\mathcal{W} = f^2 g_v^2 + h_v^2.$$

Using (2.2) we obtain the following *Gm* of the *CSF* determined by Eq. (4.1):

$$\boldsymbol{\mathcal{G}} = \frac{1}{\mathcal{W}^{1/2}} \begin{pmatrix} h_v \sin g(v) \\ -h_v \cos g(v) \\ f g_v \end{pmatrix}. \quad (4.3)$$

By taking the second derivatives w.r.t. u, v, of *CSF* described by Eq. (4.1), and by using the *Gm* given by Eq. (4.3), we find the second fundamental form matrix

$$(\mathfrak{h}_{ij}) = \frac{1}{\mathcal{W}^{1/2}} \begin{pmatrix} 0 & -f_u g_v h_v \\ -f_u g_v h_v & f(g_v h_{vv} - h_v g_{vv}) \end{pmatrix} \quad (4.4)$$

and

$$g_{vv} = \frac{\partial^2 g}{\partial v^2}, \quad h_{vv} = \frac{\partial^2 h}{\partial v^2},$$

ect. By using (4.2) and (4.4), we compute the following *SO* matrix

$$S = (\mathfrak{s}_{ij})_{2\times 2}$$

of (4.1):

$$S = \begin{pmatrix} 0 & -\frac{g_v h_v}{f_u \mathcal{W}^{1/2}} \\ -\frac{f_u g_v h_v}{\mathcal{W}^{3/2}} & \frac{f(g_v h_{vv} - h_v g_{vv})}{\mathcal{W}^{3/2}} \end{pmatrix}.$$

Finally, using (3.1), with (4.2), (4.4), respectively, we find the curvatures of the *CSF* defined by Eq. (4.1) as follows.

Theorem 3. *Let $\mathfrak{J}$ be a CSF determined by Eq.* (4.1) *in* $\mathbb{E}^3$. $\mathfrak{J}$ *contains the following curvatures*

$$\mathcal{K}_0 = 1,$$

$$2\mathcal{K}_1 = \frac{f(g_v h_{vv} - h_v g_{vv})}{\mathcal{W}^{3/2}},$$

$$\mathcal{K}_2 = -\frac{g_v^2 h_v^2}{\mathcal{W}^2}.$$

Here, $\mathcal{K}_1$ *represents the MC,* $\mathcal{K}_2$ *denotes the Gaussian curvature.*

Proof. By using the Cayley−Hamilton theorem, we reveal the following characteristic polynomial Eq. of the *SO* matrix of *CSF* defined by Eq. (4.1):

$$\mathcal{K}_0 \delta^2 - 2\mathcal{K}_1 \delta + \mathcal{K}_2 = 0$$

where

$$\mathcal{K}_0 = 1,$$

$$2\mathcal{K}_1 = \mathfrak{s}_{22},$$

$$\mathcal{K}_2 = -\mathfrak{s}_{12}\mathfrak{s}_{21}.$$

The curvatures $\mathcal{K}_i$ of $\mathfrak{J}$ are obtained by the above Eqs.

Theorem 4. *Let $\mathfrak{J}$ be a CSF described by Eq.* (4.1) *in* $\mathbb{E}^3$. $\mathfrak{J}$ *has the following principal curvatures*

$$k_{1,2} = \frac{f(g_v h_{vv} - h_v g_{vv}) \pm \left(4 g_v^2 h_v^2 (f^2 g_v^2 + h_v^2) + f^2 (g_v h_{vv} - h_v g_{vv})^2\right)^{1/2}}{2(f^2 g_v^2 + h_v^2)^{3/2}}.$$

Proof. By using Eq.

$$\det(S - k\mathcal{I}_2) = 0,$$

we have

$$k_{1,2} = \frac{1}{2}(\mathfrak{s}_{22} \pm (4\mathfrak{s}_{12}\mathfrak{s}_{21} + \mathfrak{s}_{22}{}^2)^{1/2}).$$

Then, it is clear.

Corollary 1. *Let $\mathfrak{J}$ be a CSF defined by Eq.* (4.1) *in* $\mathbb{E}^3$. $\mathfrak{J}$ *is* 1*-minimal iff the following partial differential Eq. supplies*

$$f(g_v h_{vv} - h_v g_{vv}) = 0,$$

where $\mathcal{W} \neq 0$.

Corollary 2. *Let $\mathfrak{J}$ be a CSF determined by Eq.* (4.1) *in* $\mathbb{E}^3$. $\mathfrak{J}$ *is* 2*-minimal iff the following partial differential Eq. occurs*

$$g_v^2 h_v^2 = 0,$$

where $\mathcal{W} \neq 0$.

V. Laplace−Beltrami Operator of the Conoid Surfaces Family in $\mathbb{E}^3$

In this section, our focus is on the Laplace−Beltrami operator (*LBo*) of a smooth function in $\mathbb{E}^3$. We will proceed to compute it utilizing the *CSF*, which is defined by Eq. (4.1).

Definition 6. *The LBo of a smooth function $\phi = \phi(x^1, x^2)|_{\mathcal{D}}$ $(\mathcal{D} \subset \mathbb{R}^2)$ of class C^2 is the operator defined by*

$$\Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^i}\left(\mathbf{g}^{1/2} \mathfrak{g}^{ij} \frac{\partial \phi}{\partial x^j}\right), \qquad (5.1)$$

where

$$(\mathfrak{g}^{ij}) = (\mathfrak{g}_{kl})^{-1}$$

and

$$\mathbf{g} = \det(\mathfrak{g}_{ij}).$$

Therefore, the *LBo* of the *CSF* given by Eq. (4.1) is determined by

$$\Delta\mathfrak{I} = \frac{1}{\mathbf{g}^{1/2}}\left[\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathfrak{g}^{11}\frac{\partial\mathfrak{I}}{\partial u}\right) + \frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathfrak{g}^{12}\frac{\partial\mathfrak{I}}{\partial v}\right) + \frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathfrak{g}^{21}\frac{\partial\mathfrak{I}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathfrak{g}^{22}\frac{\partial\mathfrak{I}}{\partial v}\right)\right], \qquad (5.2)$$

where

$$\mathfrak{g}^{11} = \frac{1}{f_u^2}, \ \mathfrak{g}^{12} = 0, \qquad \mathfrak{g}^{21} = 0, \ \mathfrak{g}^{22} = \frac{1}{\mathcal{W}}. \qquad (5.3)$$

Taking the derivatives of the functions determined by Eqs. (5.3) in (5.2), w.r.t. u and v, resp., we find the following.

Theorem 5. *The LBo of the CSF $\mathfrak{I}$ denoted by Eq.* (4.1) *is determined by*

$$\Delta\mathfrak{I} = 2\mathcal{K}_1\boldsymbol{\mathcal{G}},$$

where $\mathcal{K}_1$ describes the MC, $\boldsymbol{\mathcal{G}}$ represents the Gm of $\mathfrak{I}$.

Proof. With direct calculating by (5.2), we obtain

$$\Delta\mathfrak{I} = (\Delta\mathfrak{I}_1, \Delta\mathfrak{I}_2, \Delta\mathfrak{I}_3),$$

with components

$$\Delta\mathfrak{I}_1 = \frac{f h_v (g_v h_{vv} - h_v g_{vv}) \sin g(v)}{\mathcal{W}^2},$$

$$\Delta\mathfrak{I}_2 = -\frac{f h_v (g_v h_{vv} - h_v g_{vv}) \cos g(v)}{\mathcal{W}^2},$$

$$\Delta\mathfrak{I}_3 = \frac{f^2 g_v (g_v h_{vv} - h_v g_{vv})}{\mathcal{W}^2},$$

Definition 7. The surface $\mathfrak{I}$ is called harmonic if each componets of $\Delta\mathfrak{I}$ is zero.

Example 1. *Substituting*

$$f(u) = u, \ \ g(v) = v, \ \ h(v) = v$$

into a CSF defined by Eq. (4.1) *in $\mathbb{E}^3$, we have the Gm and the SO matrix, respectively,*

$$\boldsymbol{\mathcal{G}} = \frac{1}{(u^2+1)^{1/2}}\begin{pmatrix} \sin v \\ -\cos v \\ u \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & \frac{1}{(u^2+1)^{1/2}} \\ -\frac{1}{(u^2+1)^{3/2}} & 0 \end{pmatrix}.$$

The principal curvatures are given by

$$k_1 = -k_2 = \frac{1}{u^2+1},$$

and the curvatures are determined by

$$\mathcal{K}_0 = 1, \ \mathcal{K}_1 = 0, \ \mathcal{K}_2 = -\frac{1}{(u^2+1)^2}.$$

Then, we obtain

$$\Delta\mathfrak{I} = (0,0,0).$$

Finally, the surface is 1*-minimal and harmonic.*

Theorem 6. *The LBo of the CSF $\mathfrak{I}$ denoted by Eq.* (4.1) *is given by*

$$\Delta\mathfrak{I} = \mathcal{A}\mathfrak{I},$$

where

$$\mathcal{A} = \begin{pmatrix} \dfrac{h_v(g_v h_{vv} - h_v g_{vv})\tan g}{\mathcal{W}^2} & 0 & 0 \\ 0 & -\dfrac{h_v(g_v h_{vv} - h_v g_{vv})\cot g}{\mathcal{W}^2} & 0 \\ 0 & 0 & \dfrac{f^2 g_v(g_v h_{vv} - h_v g_{vv})}{h\mathcal{W}^2} \end{pmatrix}.$$

Proof. By using

$$\Delta\mathfrak{I} = 2\mathcal{K}_1\boldsymbol{\mathcal{G}},$$

it is clear.

Corollary 3. *The LBo of the CSF $\mathfrak{I}$ denoted by Eq.* (4.1) *is given by*

$$\Delta\mathfrak{I} = (0,0,0),$$

where $h\mathcal{W}^2 \neq 0$,

$$\Delta\mathfrak{I}_1 = h_v(g_v h_{vv} - h_v g_{vv})\tan g(v) = 0,$$

$$\Delta\mathfrak{I}_2 = -h_v(g_v h_{vv} - h_v g_{vv})\cot g(v) = 0,$$

$$\Delta\mathfrak{I}_3 = f^2 g_v(g_v h_{vv} - h_v g_{vv}) = 0.$$

VI. CONCLUSIONS

This paper focuses on investigating the geometric properties of the conoid surfaces family within the three-dimensional Euclidean space $\mathbb{E}^3$.

The primary objective is to analyze and comprehend the characteristics of these surfaces. The differential geometry of the conoid surfaces family plays a vital role in providing essential information about the local geometry, such as curvatures and tangent spaces. The Cayley−Hamilton theorem is employed to effectively determine the curvatures of these specific surfaces by expressing the characteristic polynomial in terms of the matrices themselves. Furthermore, the research establishes the conditions for minimality within the context of the conoid surfaces family, which serve as criteria to identify when a surface can be considered minimal in this particular family. Additionally, the exploration of the Laplace−Beltrami operator sheds light on its relationship with the conoid surfaces family.

This research contributes to an enhanced understanding of the geometric properties, curvatures, minimality conditions, and the interplay with the Laplace−Beltrami operator within the conoid surfaces family in $\mathbb{E}^3$.

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