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Conoid Surfaces Family in Minkowski 3-Space

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Abstract – This study focuses on exploring a distinct family of conoid surfaces in the three-dimensional Minkowski space $\mathbb{L}^3$. Our main objective is to delve into the differential geometry of this family, analyzing its curvatures in detail. Furthermore, we establish the essential conditions for achieving minimality within this specific framework. Additionally, we calculate the Laplace−Beltrami operator for this family of surfaces and illustrate our findings through an example.

Keywords – Minkowski 3-space, conoid surfaces family, Gauss map, curvatures, Laplace−Beltrami operator

I. INTRODUCTION

The concept of submanifolds of finite order immersed in Euclidean m-space $\mathbb{E}^m$ or pseudo-Euclidean m-space $\mathbb{E}_v^m$ was originally introduced by Chen [4, 5, 6, 7], utilizing a finite set of eigenfunctions derived from their Laplacian. Since then, this subject has been extensively examined and investigated.

Takahashi demonstrated that a Euclidean submanifold is classified as 1-type if and only if it is either minimal or minimal within a hypersphere of $\mathbb{E}^m$. The minimal submanifolds were originally provided by Lawson [20]. Subsequently, Garay [16] examined Takahashi's theorem in the context of $\mathbb{E}^m$. Aminov [2] conducted extensive research on the geometry of submanifolds. Over the course of four decades, Chen et al. [8] dedicated their research efforts to investigating 1-type submanifolds and the 1-type Gauss map (***Gm***) within the realm of space forms.

In the three-dimensional Euclidean space, denoted as $\mathbb{E}^3$, Takahashi [22] conducted an investigation into minimal surfaces. Within this framework, two types of surfaces were identified: spheres and surfaces with minimal sections. Ferrandez et al. [14] established that surfaces possessing specific characteristics can be classified as either minimal sections of a sphere or a right circular cylinder. Choi and Kim [10] directed their research towards the study of the minimal helicoid, which exhibits a pointwise 1-type ***Gm*** of the first kind. Garay [15] introduced a category of surfaces of finite type that are based on revolution. Dillen et al. [11] explored a distinct set of surfaces characterized by various properties, including minimal surfaces, spheres, and circular cylinders.

Furthermore, researchers such as Berger and Gostiaux [3], Do Carmo [12], Gray [17], and Kreyszig [18] have conducted in-depth investigations on right conoids in three-dimensional space. These studies have specifically examined different types of right conoids, including the helicoid, Whitney umbrella, Wallis's conical edge, Plücker's conoid, and hyperbolic paraboloid.

The aim of this study is to investigate the properties of the conoid surfaces family in the three-dimensional Minkowski space $\mathbb{L}^3$. Our specific objectives are to compute the matrices associated with the fundamental form, ***Gm***, and the shape operator (***so***) of this family. We will employ the Cayley−Hamilton theorem to determine the curvatures of these surfaces. Moreover, we seek to establish criteria for identifying minimality within this context. Additionally, we will explore the connection between the Laplace−Beltrami operator and these particular types of surfaces.

Section 2 provides a comprehensive overview of the fundamental principles and concepts that form the basis of three dimensional Minkowski geometry.

Section 3 is devoted to presenting the curvature formulas that are applicable to surfaces in $\mathbb{L}^3$.

In Section 4, a thorough definition of the conoid surfaces family is presented, highlighting their unique properties and characteristics.

Section 5 delves into the discussion of the Laplace−Beltrami operator for a smooth function in $\mathbb{L}^3$, and explores the utilization of the previously examined surfaces in its computation.

Finally, the research concludes in the last section.

II. Preliminaries

In this paper, the following notations, formulas, equations (Eqs.), etc., are utilized.

Consider a hypersurface M in $(n+1)$-dimensional Minkowski space $\mathbb{L}^{n+1}$, characterized by its position vector x and its ***so*** S. Let $\{e_1, e_2, \dots, e_n\}$ denote a local orthonormal frame field comprising the principal directions of M, which align with the principal curvatures k_i, where $i = 1,2, \dots, n$.

Let the dual basis of this frame field be $\{\theta_1, \theta_2, \dots, \theta_n\}$. Then, the first structural Eq. of Cartan is determined by

$$d\theta_i = \sum_{i,j=1}^{n} \theta_j \wedge \omega_{ij},$$

where ω_{ij} indicates the connection forms coinciding with the chosen frame field. By the Codazzi Eq., we derive the Eqs.

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j),$$

$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l),$$

for different $i, j, l = 1,2, \dots, n$.

We let

$$s_j = \sigma_j(k_1, k_2, \dots, k_n),$$

where σ_j denotes the j-th elementary symmetric function defined by

$$\sigma_j(a_1, a_2, \dots, a_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} a_{i_1} a_{i_2} \dots a_{i_j}.$$

We consider the notation

$$r_i^j = \sigma_j(k_1, k_2, \dots, k_{i-1}, k_{i+1}, \dots, k_n).$$

According to the given definition, we have

$$r_i^0 = 1,$$

and

$$s_{n+1} = s_{n+2} = \cdots = 0.$$

The s_k is referred to as the k-th mean curvature of the oriented hypersurface M. The mean curvature is described by

$$H = \frac{1}{n} s_1,$$

and the Gauss−Kronecker curvature of M is determined by

$$K = s_n.$$

If $s_j \equiv 0$, the hypersurface M is known as j-maximal if it is space-like, j-minimal if it is time-like hypersurface.

In Minkowski $(n+1)$-space, to obtain the curvature formulas $\mathcal{K}_i$ (See [1] and [19] for details.), $i = 0,1, \dots, n$, we have the following characteristic polynomial Eq.:

$$P_S(\lambda) = 0,$$

i.e.,

$$\sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = \det(S - \lambda)\mathcal{I}_n = 0. \quad (2.1)$$

Here, $i = 0,1, \dots, n$, $\mathcal{I}_n$ denotes the identity matrix. Hence, we reveal the curvature formulas

$$\binom{n}{i} \mathcal{K}_i = s_i.$$

Let $\mathfrak{x} = \mathfrak{x}(u, v)$ indicates an immersion from $M^2 \subset \mathbb{E}^2$ to $\mathbb{L}^3$.

Definition 1. An inner product of two vectors

$$\mathbf{v} = (v^1, v^2, v^3) \text{ and } \mathbf{w} = (w^1, w^2, w^3)$$

of $\mathbb{L}^3$ is described by

$$\langle \mathbf{v}, \mathbf{w} \rangle = v^1 w^1 + v^2 w^2 - v^3 w^3.$$

Definition 2. A vector product of

$$\mathbf{v} = (v^1, v^2, v^3) \text{ and } \mathbf{w} = (w^1, w^2, w^3)$$

of $\mathbb{L}^3$ is determined by

$$\mathbf{v} \times \mathbf{w} = \det \begin{pmatrix} e_1 & e_2 & -e_3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{pmatrix}.$$

Definition 3. The matrix

$$(\mathfrak{g}_{ij})^{-1} (\mathfrak{h}_{ij})$$

determines the ***so*** matrix S of surface $\mathfrak{x}$ in Minkowski 3-space $\mathbb{L}^3$, where

$$(\mathfrak{g}_{ij})_{2\times 2} \text{ and } (\mathfrak{h}_{ij})_{2\times 2}$$

indicate the first and the second fundamental form matrices, respectively, and

$$\mathfrak{g}_{ij} = \langle \mathfrak{x}_i, \mathfrak{x}_j \rangle, \ \ \mathfrak{h}_{ij} = \langle \mathfrak{x}_{ij}, \mathcal{G} \rangle, \ \ i, j = 1,2,$$

$\mathfrak{x}_u = \frac{\partial \mathfrak{x}}{\partial u}$ when $i = 1$, $\mathfrak{x}_{uv} = \frac{\partial^2 \mathfrak{x}}{\partial u \partial v}$ when $i = 1$ and $j = 2$, etc., e_k denotes the natural base elements of $\mathbb{E}^3$, and

$$\mathcal{G} = \frac{\mathfrak{x}_u \times \mathfrak{x}_v}{\|\mathfrak{x}_u \times \mathfrak{x}_v\|} \quad (2.2)$$

determines the ***Gm*** of the surface $\mathfrak{x}$.

III. Curvatures in Minkowski Three-Space

In this section, we obtain the curvature formulas of any surface $\mathfrak{x} = \mathfrak{x}(u, v)$ *in* $\mathbb{L}^3$.

Theorem 1. *A surface $\mathfrak{x}$ in $\mathbb{L}^3$ has the following curvature formulas,*

$$\mathcal{K}_0 = 1$$

by definition,

$$2\mathcal{K}_1 = -\frac{\mathfrak{c}_1}{\mathfrak{c}_2}, \ \ \mathcal{K}_2 = \frac{\mathfrak{c}_0}{\mathfrak{c}_2} \quad (3.1)$$

where

$$\mathfrak{c}_2 \lambda^2 + \mathfrak{c}_1 \lambda + \mathfrak{c}_0 = 0$$

*describes the characteristic polynomial Eq. of the **so** matrix,*

$$\mathfrak{c}_2 = \det(\mathfrak{g}_{ij}), \ \ \mathfrak{c}_0 = \det(\mathfrak{h}_{ij}),$$

and

$$(\mathfrak{g}_{ij})_{2\times 2} \text{ and } (\mathfrak{h}_{ij})_{2\times 2}$$

denote the first, and the second fundamental form matrices, respectively.

Proof. The matrix

$$(\mathfrak{g}_{ij})^{-1} (\mathfrak{h}_{ij})$$

describes the ***so*** matrix of surface $\mathfrak{x}$ in Minkowski 3-space $\mathbb{L}^3$. We reveal the characteristic polynomial Eq.:

$$\det(S - \lambda \mathcal{I}_2) = 0.$$

Thus, we obtain the curvatures

$$\binom{2}{0}\mathcal{K}_0 = 1,$$

$$\binom{2}{1}\mathcal{K}_1 = k_1 + k_2 = -\frac{\mathfrak{c}_1}{\mathfrak{c}_2},$$

$$\binom{2}{2}\mathcal{K}_2 = k_1 k_2 = \frac{\mathfrak{c}_0}{\mathfrak{c}_2}.$$

Definition 4. *A surface $\mathfrak{x}$ is called j-maximal (resp. j-minimal) if $\mathcal{K}_{\mathrm{j}} = 0$ and surface is space-like (resp., time-like), where $j = 1,2$.*

Theorem 2. *A surface $\mathfrak{x} = \mathfrak{x}(u, v)$ in $\mathbb{L}^3$ has the following relation*

$$\mathcal{K}_0 \mathbb{III} - 2\mathcal{K}_1 \mathbb{II} + \mathcal{K}_2 \mathbb{I} = \mathcal{O}_2,$$

where $\mathbb{I}, \mathbb{II}, \mathbb{III}$ determines the fundamental form matrices, $\mathcal{O}_2$ represents the zero matrix having order 2 of the surface.

Proof. Regarding $n = 2$ in (2.1), it runs.

IV. Conoid Surfaces Family in $\mathbb{L}^3$

In this section, we establish the definition of the conoid surfaces family (*CSF*) and subsequently explore its differential geometric properties within the Minkowski 3-space $\mathbb{L}^3$.

A ruled surface described by the Eq.

$$\begin{aligned} r(u, v) &= m(v) + u\, n(v) \\ &= (\beta(v), 0, 0) \\ &\quad + u\big(0, \cosh\alpha(v), \sinh\alpha(v)\big) \end{aligned}$$

is termed a right conoid in $\mathbb{L}^3$ if it can be generated by the translation of a straight line that intersects a fixed straight line, while ensuring that the lines maintain a perpendicular relationship throughout the generation process. By considering the yz-plane as the perpendicular plane and selecting the x-axis as the reference line, the parametric Eq. for the right conoid is given by

$$r(u, v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \beta(v) \\ u \cosh\alpha(v) \\ u \sinh\alpha(v) \end{pmatrix}.$$

The helicoid, Whitney umbrella, Wallis's conical edge, Plücker's conoid, and hyperbolic paraboloid are all instances of right conoid surfaces in $\mathbb{E}^3$. For further information, refer to the works of Berger and Gostiaux [3], Do Carmo [12], Gray [17], and Kreyszig [18].

Definition 5. *A CSF is an immersion $\mathfrak{x}$ from $M^2 \subset \mathbb{E}^2$ to $\mathbb{L}^3$ with the reference line x, defined by*

$$\mathfrak{x}(u, v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} h(v) \\ f(u) \cosh g(v) \\ f(u) \sinh g(v) \end{pmatrix} \tag{4.1}$$

where

$$f = f(u), \quad g = g(v), \quad h = h(v)$$

denote the differentiable functions.

Taking the first derivatives of *CSF* $\mathfrak{x}$ determined by Eq. (4.1), w.r.t. u, v, respectively, we obtain the first fundamental form matrix

$$(\mathfrak{g}_{ij}) = \begin{pmatrix} f_u^2 & 0 \\ 0 & -f^2 g_v^2 + h_v^2 \end{pmatrix} \tag{4.2}$$

and

$$f_u^2 = \left(\frac{\partial f}{\partial u}\right)^2, \quad g_v^2 = \left(\frac{\partial g}{\partial v}\right)^2, \quad h_v^2 = \left(\frac{\partial h}{\partial v}\right)^2.$$

Hence,

$$\det(\mathfrak{g}_{ij}) = \varepsilon f_u^2 \mathcal{Q},$$

where $\varepsilon = -1$,

$$\mathcal{Q} = f^2 g_v^2 - h_v^2.$$

Using (2.2) we obtain the following ***Gm*** of the *CSF* determined by Eq. (4.1):

$$\mathcal{G} = \frac{1}{Q^{1/2}}\begin{pmatrix} f g_v \\ h_v \sinh g(v) \\ h_v \cosh g(v) \end{pmatrix}. \qquad (4.3)$$

Taking the second derivatives w.r.t. u, v, of *CSF* described by Eq. (4.1), and by using the ***Gm*** given by Eq. (4.3), we find the second fundamental form matrix

$$(\mathfrak{h}_{ij}) = \frac{1}{Q^{1/2}}\begin{pmatrix} 0 & -f_u g_v h_v \\ -f_u g_v h_v & f(g_v h_{vv} - h_v g_{vv}) \end{pmatrix} \quad (4.4)$$

and

$$g_{vv} = \frac{\partial^2 g}{\partial v^2}, \quad h_{vv} = \frac{\partial^2 h}{\partial v^2},$$

ect. By using (4.2) and (4.4), we compute the following ***so*** matrix

$$S = (\mathfrak{s}_{ij})_{2\times 2}$$

of (4.1):

$$S = \begin{pmatrix} 0 & -\dfrac{g_v h_v}{f_u Q^{1/2}} \\ \dfrac{f_u g_v h_v}{Q^{3/2}} & \dfrac{f(g_v h_{vv} - h_v g_{vv})}{Q^{3/2}} \end{pmatrix}.$$

Hence, using (3.1), with (4.2), (4.4), respectively, we find the curvatures of the *CSF* defined by Eq. (4.1) as follows.

Theorem 3. *Let* $\mathfrak{x}$ *be a CSF determined by Eq.* (4.1) *in* $\mathbb{L}^3$. $\mathfrak{x}$ *contains the following curvatures*

$$\mathcal{K}_0 = 1,$$

$$2\mathcal{K}_1 = \frac{f(g_v h_{vv} - h_v g_{vv})}{Q^{3/2}},$$

$$\mathcal{K}_2 = \frac{g_v^2 h_v^2}{Q^2}.$$

Here, $\mathcal{K}_1$ *represents the mean curvature,* $\mathcal{K}_2$ *denotes the Gaussian curvature.*

Proof. By the Cayley−Hamilton theorem, we reveal the following characteristic polynomial Eq. of the ***so*** matrix of *CSF* defined by Eq. (4.1):

$$\mathcal{K}_0\mu^2 - 2\mathcal{K}_1\mu + \mathcal{K}_2 = 0,$$

where

$$\mathcal{K}_0 = 1,$$

$$2\mathcal{K}_1 = \mathfrak{s}_{22},$$

$$\mathcal{K}_2 = -\mathfrak{s}_{12}\mathfrak{s}_{21}.$$

The curvatures $\mathcal{K}_i$ of $\mathfrak{x}$ are obtained by the above Eqs.

Theorem 4. *Let* $\mathfrak{x}$ *be a CSF described by Eq.* (4.1) *in* $\mathbb{L}^3$. $\mathfrak{x}$ *has the following principal curvatures*

$$k_{1,2} = \frac{f(g_v h_{vv} - h_v g_{vv}) \pm \left(-4g_v^2 h_v^2 (f^2 g_v^2 - h_v^2) + f^2 (g_v h_{vv} - h_v g_{vv})^2\right)^{1/2}}{2(f^2 g_v^2 - h_v^2)^{3/2}}.$$

Proof. By using Eq.

$$\det(S - k\mathcal{I}_2) = 0,$$

we have

$$k_{1,2} = \frac{1}{2}(\mathfrak{s}_{22} \pm (4\mathfrak{s}_{12}\mathfrak{s}_{21} + \mathfrak{s}_{22}{}^2)^{1/2}).$$

Then, it is obvious.

Corollary 1. *Let* $\mathfrak{x}$ *be a CSF defined by Eq.* (4.1) *in* $\mathbb{L}^3$. $\mathfrak{x}$ *is* 1*-minimal iff the following partial differential Eq. appears*

$$f(g_v h_{vv} - h_v g_{vv}) = 0,$$

where $Q \neq 0$.

Corollary 2. *Let $\mathfrak{x}$ be a CSF determined by Eq.* (4.1) *in* $\mathbb{L}^3$. $\mathfrak{x}$ *is* 2*-minimal iff the following partial differential Eq. occurs*

$$g_v^2 h_v^2 = 0,$$

where $\mathcal{Q} \neq 0$.

V. LAPLACE–BELTRAMI OPERATOR OF THE CONOID SURFACES FAMILY IN $\mathbb{L}^3$

In this section, our focus is on the Laplace–Beltrami operator (*LBo*) of a smooth function in $\mathbb{L}^3$. We will proceed to compute it utilizing the *CSF*, which is defined by Eq. (4.1).

Definition 6. *The LBo of a smooth function* $\phi = \phi(x^1, x^2)$ $(\mathcal{D} \subset \mathbb{R}^2)$ *of class* C^2 *is the operator defined by*

$$\Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^i}\left(\mathbf{g}^{1/2} \mathrm{g}^{ij} \frac{\partial \phi}{\partial x^j}\right), \quad (5.1)$$

where

$$(\mathrm{g}^{ij}) = (\mathrm{g}_{kl})^{-1}$$

and

$$\mathbf{g} = \det(\mathrm{g}_{ij}).$$

Therefore, the *LBo* of the *CSF* given by Eq. (4.1) is determined by

$$\Delta\mathfrak{x} = \frac{1}{\mathbf{g}^{1/2}}\left[\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathrm{g}^{11}\frac{\partial \mathfrak{x}}{\partial u}\right) + \left[\frac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathrm{g}^{12}\frac{\partial \mathfrak{x}}{\partial v}\right)\right.\right.$$
$$\left. + \frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathrm{g}^{21}\frac{\partial \mathfrak{x}}{\partial u}\right) + \left[\frac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathrm{g}^{22}\frac{\partial \mathfrak{x}}{\partial v}\right)\right]\right] \quad (5.2)$$

where

$$\mathrm{g}^{11} = \frac{1}{f_u^2}, \ \mathrm{g}^{12} = 0,$$
$$\mathrm{g}^{21} = 0, \ \mathrm{g}^{22} = \frac{1}{\mathcal{Q}}. \quad (5.3)$$

By taking the derivatives of the functions determined by Eqs. (5.3) in (5.2), w.r.t. u and v, resp., we determine the following.

Theorem 5. *The LBo of the CSF* $\mathfrak{x}$ *denoted by Eq.* (4.1) *in* $\mathbb{L}^3$ *is given by*

$$\Delta\mathfrak{x} = 2\mathcal{K}_1\mathcal{G},$$

where $\mathcal{K}_1$ *describes the mean curvature,* $\mathcal{G}$ *represents the* ***Gm*** *of* $\mathfrak{x}$.

Proof. Via direct calculating by (5.2), we obtain

$$\Delta\mathfrak{x} = (\Delta\mathfrak{x}_1, \Delta\mathfrak{x}_2, \Delta\mathfrak{x}_3),$$

with components

$$\Delta\mathfrak{x}_1 = \frac{f^2 g_v\left(g_v h_{vv} - h_v g_{vv}\right)}{\mathcal{Q}^2},$$

$$\Delta\mathfrak{x}_2 = \frac{f h_v\left(g_v h_{vv} - h_v g_{vv}\right)\sinh g(v)}{\mathcal{Q}^2},$$

$$\Delta\mathfrak{x}_3 = \frac{f h_v\left(g_v h_{vv} - h_v g_{vv}\right)\cosh g(v)}{\mathcal{Q}^2}.$$

Definition 7. *The surface* $\mathfrak{x}$ in $\mathbb{L}^3$ *is called harmonic if each componets of* $\Delta\mathfrak{x}$ *is zero*.

Example 1. *Substituting*

$$f(u) = u, \ g(v) = v, \ h(v) = v$$

into a CSF $\mathfrak{x}$ *defined by Eq.* (4.1) *in* $\mathbb{L}^3$, *the curvatures are determined by*

$$\mathcal{K}_0 = 1, \ \mathcal{K}_1 = 0, \ \mathcal{K}_2 = \frac{1}{(u^2-1)^2}.$$

Then, the surface is 1*-maximal if* $\mathfrak{x}$ *is space-like,* 1*-minimal if* $\mathfrak{x}$ *is time-like. Therefore,*

$$\Delta\mathfrak{x} = (0,0,0).$$

Finally, $\mathfrak{x}$ *is a harmonic surface.*

Theorem 6. *The LBo of the CSF $\mathfrak{x}$ denoted by Eq.* (4.1) *in* $\mathbb{L}^3$ *is given by*

$$\Delta\mathfrak{x} = \mathcal{B}\mathfrak{x},$$

where

$$\mathcal{B} = \begin{pmatrix} \dfrac{f^2 g_v\left(g_v h_{vv} - h_v g_{vv}\right)}{hQ^2} & 0 & 0 \\ 0 & \dfrac{h_v\left(g_v h_{vv} - h_v g_{vv}\right)\tanh g(v)}{Q^2} & 0 \\ 0 & 0 & \dfrac{h_v\left(g_v h_{vv} - h_v g_{vv}\right)\coth g(v)}{Q^2} \end{pmatrix}.$$

Proof. Taking care of

$$\Delta\mathfrak{x} = 2\mathcal{K}_1\mathcal{G},$$

it is obvious.

Corollary 3. *The LBo of the CSF $\mathfrak{x}$ denoted by Eq.* (4.1) *in* $\mathbb{L}^3$ *is given by*

$$\Delta\mathfrak{x} = (0,0,0),$$

where

$$\Delta\mathfrak{x}_1 = f^2 g_v\left(g_v h_{vv} - h_v g_{vv}\right) = 0,\ \ hQ^2 \neq 0,$$

$$\Delta\mathfrak{x}_2 = h_v\left(g_v h_{vv} - h_v g_{vv}\right)\tanh g(v) = 0,\ Q^2 \neq 0,$$

$$\Delta\mathfrak{x}_3 = h_v(g_v h_{vv} - h_v g_{vv})\coth g(v) = 0, Q^2 \neq 0.$$

Proof. If $\mathcal{B} = 0$ in Theorem 6, then it is clear.

VI. Conclusions

This paper aims to explore the geometric characteristics of the conoid surfaces family in the three-dimensional Minkowski space $\mathbb{L}^3$.

The differential geometry of the conoid surfaces family plays a crucial role in providing important insights into the local geometry, including curvatures and tangent spaces. The utilization of the Cayley−Hamilton theorem effectively determines the curvatures of these specific surfaces by expressing the characteristic polynomial in terms of the matrices themselves. Moreover, this study establishes the conditions for minimality within the conoid surfaces family, serving as criteria to determine when a surface can be considered minimal in this particular family. Furthermore, investigating the Laplace−Beltrami operator sheds light on its relationship with the conoid surfaces family.

This research significantly enhances our understanding of the geometric properties, curvatures, conditions for minimality, and the relationship with the Laplace−Beltrami operator in the context of the conoid surfaces family in $\mathbb{L}^3$.

References

[1] L. J. Alias and N. Gürbüz: *An extension of Takashi theorem for the linearized operators of the highest order mean curvatures*. Geom. Dedicata **121** (2006), 113-127.

[2] Y. Aminov: *The Geometry of Submanifolds*. Gordon and Breach Sci. Pub., Amsterdam, 2001.

[3] M. Berger and B. Gostiaux: *Géométrie Différentielle: Variétés, Courbes et Surfaces*. Presses Univ. France, France, 1987.

[4] B. Y. Chen: *On submanifolds of finite type*. Soochow J. Math. **9** (1983), 65-81.

[5] B. Y. Chen: *Total Mean Curvature and Submanifolds of Finite Type*. World Scientific, Singapore, 1984.

[6] B. Y. Chen: *Finite Type Submanifolds and Generalizations*. University of Rome, Rome, 1985.

[7] B. Y. Chen: *Finite type submanifolds in pseudo-Euclidean spaces and applications*. Kodai Math. J. **8**(3) (1985), 358-374.

[8] B. Y. Chen, E. Güler, Y. Yaylı, and H. H. Hacısalihoğlu: *Differential geometry of 1-type submanifolds and submanifolds with 1-type Gauss map*. Int. Elec. J. Geom., **16**(1) (2023), 4-49.

[9] B. Y. Chen and P. Piccinni: *Submanifolds with finite type Gauss map*. Bull. Aust. Math. Soc. **35** (1987), 161-186.

[10] M. Choi and Y. H. Kim: *Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map*. Bull. Korean Math. Soc. **38** (2001), 753-761.

[11] F. Dillen, J. Pas, and L. Verstraelen: *On surfaces of finite type in Euclidean 3-space*. Kodai Math. J. **13** (1990), 10-21.

[12] M. P. Do Carmo: *Differential Geometry of Curves and Surfaces*. Englewood Cliffs, Prentice Hall, USA, 1976.

[13] M. P. Do Carmo and M. Dajczer: *Rotation hypersurfaces in spaces of constant curvature*. Trans. Amer. Math. Soc. **277** (1983), 685-709.

[14] A. Ferrandez, O. J. Garay, and P. Lucas: *On a certain class of conformally at Euclidean hypersurfaces*. In: Proceedings of a Conference on Global Analysis and Global Differential Geometry. Springer, Berlin, 1990, pp. 48-54.

[15] O. J. Garay: *On a certain class of finite type surfaces of revolution*. Kodai Math. J. **11** (1988), 25-31.

[16] O. J. Garay: *An extension of Takahashi's theorem*. Geom. Dedicata **34** (1990), 105-112.

[17] A. Gray: *Modern Differential Geometry of Curves and Surfaces with Mathematica*. 2nd ed. Boca Raton, FL: CRC Press, 1997.

[18] A. Kreyszig: *Differential Geometry*. Dover Pub., New York, 1991.

[19] W. Kühnel: *Differential Geometry. Curves-Surfaces-Manifolds*. Third ed. Translated from the 2013 German ed. AMS, Providence, RI, 2015.

[20] H. B. Lawson: *Lectures on Minimal Submanifolds*, 2nd ed.; Mathematics Lecture Series 9; Publish or Perish, Inc.: Wilmington, DE, USA, 1980.

[21] T. Levi−Civita: *Famiglie di superficie isoparametriche nellordinario spacio euclideo.* Rend. Acad. Lincei **26** (1937), 355-362.

[22] T. Takahashi: *Minimal immersions of Riemannian manifolds*. J. Math. Soc. Japan **18** (1966), 380-385.