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Some Observations on g-Metric Spaces in Light of Generalized Statistical Convergence of Double Sequences via Ideals

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Abstract – In this article, we delve into the notions of $\mathcal{J}_2$-statistical convergence and $\mathcal{J}_2$-lacunary statistical convergence for sequences in general metric spaces, specifically g metric spaces. We thoroughly explore these concepts within the realm of g-metric spaces.

Keywords – Statistical Convergence, Lacunary Sequence, Ideal Convergence, g-Metric Space

I. Introduction

The formal introduction of statistical convergence was pioneered by Fast [1], who built upon the concept of natural density. The definition of the natural density of a set $K \subseteq \mathbb{N}$ is as follows:

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{i \in K: i \leq n\}|,$$

where the vertical bars denote the cardinality of the set. A sequence of real numbers $u = (u_i)$ is considered to be statistically convergent to $u_0 \in \mathbb{R}$ if, for any given $\sigma > 0$, the following condition is satisfied:

$$\delta(\{i \in \mathbb{N}: |u_i - u_0| \geq \sigma\}) = 0.$$

The concept introduced by Mursaleen and Edely [2] was further expanded to encompass double sequences. In the existing literature, researchers such as Connor [3], Tripathy [4], and many others have explored statistical convergence.

In a separate development, Fridy and Orhan [5] introduced an extension of statistical convergence known as lacunary statistical convergence. The definition of lacunary statistical convergence is outlined below:

A lacunary sequence is an increasing integer sequence $\theta = (k_n)_{n\in\mathbb{N}\cup\{0\}}$ satisfying $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$, as $n \to \infty$.

A sequence of real numbers $u = (u_i)$ is defined to be lacunary statistically convergent to $u_0 \in \mathbb{R}$ if, for any given $\sigma > 0$, the following condition is met:

$$\lim_{n\to\infty} \frac{1}{h_n} |\{i \in I_n: |u_i - u_0| \geq \sigma\}| = 0$$

where $I_n = (k_{n-1}, k_n]$.

In an effort to generalize the concept of statistical convergence, Kostyrko et al. [6] put forward the notions of $\mathcal{J}$ and $\mathcal{J}^*$-convergence. These concepts involve the use of an ideal, denoted as $\mathcal{J}$, which represents a family of subsets of a non-empty set X. The ideal $\mathcal{J}$ satisfies closure properties under finite unions and subsets of its elements. A non-trivial ideal $\mathcal{J}$ is characterized by $\mathcal{J} \neq \emptyset$ and $X \notin \mathcal{J}$. Furthermore, an admissible ideal in X is a non-trivial ideal $\mathcal{J} \subset P(X)$ that includes the singleton sets $\mathcal{J} \supset \{\{w\}: w \in X\}$.

It can be easily verified that $\mathcal{J}_f = \{A \subseteq \mathbb{N}: |A| < \infty\}$, $\mathcal{J}_\delta = \{A \subseteq \mathbb{N}: \delta(A) = 0\}, \mathcal{J}_c = \{A \subseteq \mathbb{N}: \sum_{a\in A} a^{-1} < \infty\}$, and $\mathcal{J} = \{A \subseteq \mathbb{N}: |\{p \in \mathbb{N}: A \cap D_p \neq \emptyset\}| < \infty\}$, where $\mathbb{N} =$

$\bigcup_{p=1}^{\infty} D_p$ is a disjoint decomposition of $\mathbb{N}$ forms an admissible ideal in $\mathbb{N}$.

In set theory, a filter $\mathcal{F}$ on a non-empty set X is a family of subsets of X that satisfies closure under finite intersections and supersets. If $\mathcal{I}$ is an ideal, the corresponding filter $\mathcal{F}(\mathcal{I})$, defined as $\{X \setminus A: A \in \mathcal{I}\}$, is known as the dual filter or filter associated with the ideal $\mathcal{I}$.

Consider a non-trivial ideal $\mathcal{I} \subset P(\mathbb{N})$ defined on the set of natural numbers, N. In the context of this ideal, a sequence of real numbers $u = (u_i)$ is considered $\mathcal{I}$-convergent to $u_0 \in \mathbb{R}$ provided that for each $\sigma > 0$, the following condition holds:

$$\{i \in \mathbb{N}: |u_i - u_0| \geq \sigma\} \in \mathcal{I}.$$

This type of convergence is symbolically denoted as $\mathcal{I} - \lim_{i\to\infty} u_i = u_0$.

Notably, when considering specific ideals, namely or $\mathcal{I} = \mathcal{I}_f$ and $\mathcal{I} = \mathcal{I}_\delta$, the above definition simplifies to the familiar definitions of usual convergence and statistical convergence, respectively.

In the field of mathematical analysis, the concept of distance functions, or metrics, serves as a generalization of physical distance. Various approaches have been developed to extend the notion of distance functions [7]. Due to the challenges posed by massive and intricate data sets, a generalized description of distance functions has become necessary. Gähler [8] introduced the idea of a 2-metric as a generalization of the conventional metric. However, subsequent research has shown that there is no direct relationship between these two types of functions. For instance, Ha et al. [9] demonstrated that a 2-metric does not necessarily exhibit continuity with respect to its variables.

Building upon these findings, Bapure Dhage [10] conducted an in-depth investigation into a new class of generalized metric spaces known as D-metric spaces, aiming to establish topological properties within these spaces. Dhage's work laid the foundation for further studies in this field. However, subsequent research, as mentioned in [11,12], has indicated that many claims regarding the basic topological features of D-metric spaces are incorrect, thereby invalidating numerous results obtained within these spaces.

Among these various generalizations, an alternative extension of the traditional metric is the concept of G-metric space developed by Mustafa and Sims [13]. Distances between three points form the metrics within this space. Notably, obtaining α within the interior of a triangle signifies that the perimeter of a triangle with vertices $x, y,$ and z in $\mathbb{R}^2$, denoted as $G(x; y; z)$, is best represented by the property $(G5)$. The G-metric function generalizes the notion of distance between three points. Choi et al. [14] extended the study to g-metrics of degree n, which involves the distance between $n + 1$ points, aiming to achieve greater generality. In their work, they investigated the extension of sequence convergence to ideal forms based on the topological features of g-metric spaces. Abazari [15] introduced statistically convergent sequences with respect to the metrics on g-metric spaces and explored fundamental properties of this statistical form of convergence.

This research unveils a novel form of convergence for sequences within g-metric spaces. The study is organized as follows: Section 1 offers an extensive literature review, while Section 2 presents the principal findings. To establish the fundamental properties of these concepts, our analysis focuses on the notions of $\mathcal{I}_2$-statistical convergence and $\mathcal{I}_2$-lacunary statistical convergence for sequences in g-metric spaces.

II. Definitions and Basic Properties

The purpose of this section is to bring together the essential data and methods needed to achieve our main objectives. We will start by introducing several crucial terms.

Definition 2.1. Let Y be a nonempty set. A function $G: Y \times Y \times Y \to \mathbb{R}^+$ is called a generalized metric, or G-metric, on Y if it satisfies the following five properties:

G1. $G(u, v, w) = 0$ iff $u = v = w$,

G2. $0 < G(u, u, v)$; for each $u, v \in G$, with $u \neq v$,

G3. $G(u, u, v) \leq G(u, v, w)$, for each $u, v, w \in Y$ with $w \neq v$,

G4. $G(u, v, w) = G(u, w, v) = G(v, w, u) = \cdots$ (symmetry in all three variables),

G5. $G(u, v, w) \leq G(u, \alpha, \alpha) + G(\alpha, v, w)$, for each $u, v, w, \alpha \in Y$ (rectangle inequality).

The pair (Y, G) is referred to as a G-metric space.

In a more scientifically precise manner, Choi et al. [14] proposed the introduction of g-metric functions distinguished by their degree, represented as n. These functions essentially define a distance measure between $n+1$ points in a given space. To precisely define this concept, we present the subsequent definition, which describes the notion of a g-metric space with a specific degree, denoted as p.

Definition 2.2. Suppose Y be a nonempty set. A function $g: Y^{p+1} \to \mathbb{R}^{+}$that supplies the following features is named g-metric with order p on Y. (Y, g) is named as a g-metric space.

(i) $g(w_0, w_1, \dots, w_p) = 0$ iff $w_0 = w_1 = \cdots = w_p$,

(ii) $g(w_0, w_1, \dots, w_p) = g(w_{\rho(0)}, w_{\rho(1)}, \dots, w_{\rho(p)})$, for permutation ρ on $\{0,1, \dots, p\}$,

(iii) $g(w_0, w_1, \dots, w_p) \leq g(q_0, q_1, \dots, q_p)$, for each $(w_0, w_1, \dots, w_p), (q_0, q_1, \dots, q_p) \in Y^{p+1}$ with

$$\{w_i : i = 0,1, \dots, p\} \subseteq \{q_i : i = 0,1, \dots, p\}$$

(iv) For all $w_0, w_1, \dots, w_s, q_0, q_1, \dots, q_t, v \in Y$ with $s + t + 1 = p$

$$g(w_0, w_1, \dots, w_s, q_0, q_1, \dots, q_t) \leq g(w_0, w_1, \dots, w_s, v, v, \dots, v) + g(q_0, q_1, \dots, q_t, v, v, \dots, v).$$

It is obvious that when $p = 1$ we have ordinary metric space and when $p = 2$ we have G-metric space.

Theorem 2.1. Let Y be a nonempty set and g be a metric on Y with order p. In this context, the following are provided:

(i)

$$g(\underbrace{w, \dots, w}_{\text{s times}}, q, \dots, q) \leq g(\underbrace{w, \dots, w}_{\text{s times}}, u, \dots, u) + g(\underbrace{u, \dots, u}_{\text{s times}}, q, \dots, q)$$

(ii)

$$g(w, q, \dots, q) \leq g(w, u, \dots, u) + g(u, q, \dots, q),$$

(iii)

$$g(\underbrace{w, \dots, w}_{\text{s times}}, u, \dots, u) \leq s g(w, u, \dots, u)$$

and

$$g\left(\underbrace{w, \dots, w}_{s \text{ times}}, u, \dots, u\right) \leq (p + 1 - s) g(u, w, \dots, w),$$

(iv)

$$g(w_0, w_1, \dots, w_p) \leq \sum_{i=0}^{n} g(w_i, u, \dots, u),$$

(v)

$$\left|g(q, w_1, w_2, \dots., w_p) - g(u, w_1, w_2, \dots., w_p)\right| \leq \max\{g(q, u, \dots, u), g(u, q, \dots, q)\},$$

(vi)

$$\left|g(\underbrace{w, \dots, w}_{\text{s times}}, u, \dots, u) - g(\underbrace{w, \dots, w}_{s' \text{ times}}, u, \dots, u)\right| \leq |s - s'| g(w, u, \dots, u),$$

(vii)

$$g(w, u, \dots, u) \leq (1 + (s - 1))(p + 1 - s) g(\underbrace{w, \dots, w}_{\text{s times}}, u, \dots, u).$$

Definition 2.3. Suppose (Y, g) be a g-metric space, $w \in Y$ be a point and $(w_i) \in Y$.

(i) (w_i) is g-convergent to w, provided for all $\sigma > 0$, there exists $N \in \mathbb{N}$ so that for $i_1, i_2, \dots, i_p \geq N$

$$g(w, w_1, w_2, \dots, w_p) < \sigma$$

(ii) (w_i) is called to be g-Cauchy, provided for all $\sigma > 0$, there exists $N \in \mathbb{N}$ so that

$$i_0, i_1, i_2, \dots, i_p \geq N \Rightarrow g\left(w_{i_0}, w_{i_1}, w_{i_2}, \dots, w_{i_p}\right) < \sigma$$

Definition 2.4. Take $p \in \mathbb{N}, K \in \mathbb{N}^p$ and

$$K(m) = \{i_1, i_2, \dots, i_p \le m, (i_1, i_2, \dots, i_p) \in \mathbb{N}^p\}$$

then

$$D_p(K) := \lim_{m\to\infty} \frac{p!}{m^p} |K(m)|,$$

is called p-dimensional natural (or asymptotic) density of the set K.

Definition 2.5. Assume (w_i) be a sequence in a g-metric space (Y, g).

(i) (w_i) is statistically convergent to w, provided for all $\sigma > 0$,

$$\lim_{m\to\infty} \frac{p!}{m^p} \left| \left\{ \begin{matrix} (i_1, i_2, \dots, i_p) \in \mathbb{N}^p \\ i_1, i_2, \dots, i_p \le m, g\left(w, w_{i_1}, w_{i_2}, \dots, w_{i_p}\right) \ge \sigma \end{matrix} \right\} \right| = 0,$$

and is indicated by $gS - \lim_{m\to\infty} w_i = w$.

(ii) (w_i) is called to be statistical g-Cauchy, provided for all $\sigma > 0$, there exists $i_\sigma \in \mathbb{N}$ such that

$$\lim_{m\to\infty} \frac{p!}{m^p} \left| \left\{ \begin{matrix} (i_1, i_2, \dots, i_p) \in \mathbb{N}^p \\ i_1, i_2, \dots, i_p \le m, g\left(w_{i_\sigma}, w_{i_1}, w_{i_2}, \dots, w_{i_p}\right) \ge \sigma \end{matrix} \right\} \right| = 0.$$

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is obvious that a strongly admissible ideal is admissible also.

Definition 2.6. Assume (Y, ρ) be a metric space A sequence $u = (u_{jk})$ in Y is called to be $\mathcal{I}_2$-convergent to u_0, provided that for any $\sigma > 0$ we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N}: \rho(u_{jk}, u_0) \ge \varepsilon\} \in \mathcal{I}_2,$$

and indicated as $\mathcal{I}_2 - \lim_{j,k\to\infty} u_{jk} = u_0$.

Definition 2.7. A sequence $u = (u_{jk})$ is $\mathcal{I}_2$-statistically convergent to u_0, provided that for any $\sigma, \delta > 0$

$$\left\{(n, m) \in \mathbb{N}^2: \frac{1}{nm} \left|\{(j, k), j \le n, k \le m: \rho(u_{jk}, u_0) \ge \sigma\}\right| \ge \delta \right\} \in \mathcal{I}_2,$$

and illustrated as $\mathcal{I}_2 - st\lim_{j,k\to\infty} u_{jk} = u_0$.

A double lacunary sequence, denoted as $\theta_2 = \theta_{r,s} = \{(k_r, l_s)\}$ is defined by the existence of two increasing sequences of integers (j_r) and (k_s), satisfying the subsequent conditions:

$$j_0 = 0, h_r = j_r - j_{r-1} \to \infty \text{ and } k_0 = 0, h_s = k_s - k_{s-1} \to \infty,\ r, s \to \infty$$

We will utilize the following notations $k_{rs} := j_r k_s, h_{rs} := h_r h_s$ and θ_{rs} is given by

$$I_{rs} := \{(j, k): j_{r-1} < j \le j_r \text{ and } k_{s-1} < k \le k_s\},$$
$$q_r := \frac{j_r}{j_{r-1}}, q_s := \frac{k_s}{k_{s-1}} \text{ and } q_{rs} := q_r q_s.$$

Throughout the paper, by $\theta_2 = \theta_{r,s} = \{(j_r, k_s)\}$ we will denote a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

III. Main Results

Before we state the main results of this work, let us give the definition of a new statistical method.

Definition 3.1. A sequence (u_{jk}) is defined as g-lacunary statistically convergent to u if, for all $\sigma > 0$,

$$\lim_{r\to\infty} \frac{p!}{(h_{rs})^p} \left|\left\{(j_w, k_w) \in I_{rs}, 1 \le w \le p: g\left(u, u_{j_1 k_1}, \cdots, u_{j_p k_p}\right) \ge \sigma\right\}\right| = 0,$$

and is indicated by $gS_{\theta_2} - \lim_{j,k\to\infty} u_{jk} = u$.

Definition 3.2. The sequence (u_{jk}) is considered to be $g-\mathcal{J}_2$-convergent to u provided that for all $\sigma>0$,

$$\left\{\left((j_1,\cdots,j_p),(k_1,\cdots,k_p)\right)\in\mathbb{N}^p\times\mathbb{N}^p: g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\in\mathcal{J}_2,$$

and is demonstrated by $g\mathcal{J}_2-\lim_{j,k\to\infty}u_{jk}=u$.

Definition 3.3. The sequence (u_{jk}) is called to be g-$\mathcal{J}_2$-statistically convergent to u if, for all $\sigma,\delta>0$,

$$\left\{(n,m)\in\mathbb{N}^2:\frac{p!}{(nm)^p}\,|\left\{\left((j_1,\cdots,j_p),(k_1,\cdots,k_p)\right)\in\mathbb{N}^p\times\mathbb{N}^p\right.\right.$$
$$\left.\left.j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m: g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}|\geq\delta\right\}\in\mathcal{J}_2,$$

and is indicated by $gS_{\mathcal{J}_2}-\lim_{j,k\to\infty}u_{jk}=u$.

Definition 3.4. The sequence (u_{jk}) is named to be g - $\mathcal{J}_2$-lacunary statistically convergent to u provided that for all $\sigma,\delta>0$,

$$\left\{(r,s)\in\mathbb{N}^2:\frac{p!}{(h_{rs})^p}\left|\left\{(j_w,k_w)\in I_{rs},1\leq w\leq p: g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\right|\geq\delta\right\}\in\mathcal{J}_2,$$

and is indicated by $gS_{\theta_2}(\mathcal{J}_2)-\lim_{j,k\to\infty}u_{jk}=u$.

Definition 3.5. The sequence (u_{jk}) is called to be g-strongly $\mathcal{J}_2$-lacunary convergent to u provided that for all $\sigma>0$,

$$\left\{(r,s)\in\mathbb{N}^2:\frac{p!}{(h_{rs})^p}\sum_{(j_w,k_w)\in I_{rs},1\leq w\leq p} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\in\mathcal{J}_2$$

and is indicated by, $gN_{\theta_2}[\mathcal{J}_2]-\lim_{j,k\to\infty}u_{jk}=u$.

Definition 3.6. The sequence (u_{jk}) is named to be g-strongly $\mathcal{J}_2$-Cesàro summable to u provided that for all $\sigma>0$,

$$\left\{(n,m)\in\mathbb{N}^2:\frac{p!}{(nm)^p}\sum_{j_1,\cdots,j_p=1}^{n}\sum_{k_1,\cdots,k_p=1}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\in\mathcal{J}_2$$

and is indicated by, $g[C,1,1][\mathcal{J}_2]-\lim_{j,k\to\infty}u_{jk}=u$.

The subsequent theorem establishes the connection between $g[C,1,1][\mathcal{J}_2]$ and $gS_{\mathcal{J}_2}$.

Theorem 3.7. Assume (Y,g) be a g-metric space, $(u_{jk})\in Y$.

(i) $g[C,1,1][\mathcal{J}_2]-\lim_{j,k\to\infty}u_{jk}=u$ implies $gS_{\mathcal{J}_2}-\lim_{j,k\to\infty}u_{jk}=u$.

(ii) If g is a bounded function, $gS_{\mathcal{J}_2}-\lim_{j,k\to\infty}u_{jk}=u$ implies $[C,1,1][\mathcal{J}_2]-\lim_{j,k\to\infty}u_{jk}=u$.

Proof. (i) Take $\sigma>0$ and $g[C,1,1][\mathcal{J}_2]-\lim_{j,k\to\infty}u_{jk}=u$. At that time, we have

$$\frac{p!}{(nm)^p}\sum_{j_1,\cdots,j_p=1}^{n}\sum_{k_1,\cdots,k_p=1}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)$$
$$\geq \frac{p!}{(nm)^p}\sum_{\substack{j_1,\cdots,j_p=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma}}^{n}\sum_{\substack{k_1,\cdots,k_p=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma}}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)$$
$$\geq \sigma\frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m: g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\right|$$

and so

$$\frac{p!}{\sigma(nm)^p}\sum_{j_1,\cdots,j_p=1}^{n}\sum_{k_1,\cdots,k_p=1}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)$$
$$\geq \frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m: g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\right|.$$

Then, for any $\delta>0$

$$\left\{(n,m)\in\mathbb{N}^2:\frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m, g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\right|\geq\delta\right\}$$
$$\subseteq\left\{(n,m)\in\mathbb{N}^2:\frac{p!}{(nm)^p}\sum_{j_1,\cdots,j_p=1}^{n}\sum_{k_1,\cdots,k_p=1}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\delta\right\}.$$

As $g[C,1,1][\mathcal{I}_2]-\lim_{j,k\to\infty}u_{jk}=u$, it can be inferred that the set on the right-hand side belongs to $\mathcal{I}_2$, leading to the conclusion that $gS_{\mathcal{I}_2}-\lim_{j,k\to\infty}u_{jk}=u$.

(ii) Now, presume that $gS_{\mathcal{I}_2}-\lim_{j,k\to\infty}u_{jk}=u$ and $\sigma>0$ given. From the boundedness of g, there exists $Q>0$ such that $g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\leq Q$ for all $j_1,j_2,\cdots,j_p$ and $k_1,k_2,\cdots,k_p$. Then

$$\frac{p!}{(nm)^p}\sum_{j_1,\cdots,j_p=1}^{n}\sum_{k_1,\cdots,k_p=1}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)$$
$$=\frac{p!}{(nm)^p}\sum_{\substack{j_1,\cdots,j_p=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma}}^{n}\sum_{\substack{k_1,\cdots,k_p=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma}}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)$$
$$+\frac{p!}{(nm)^p}\sum_{\substack{j_1,\cdots,j_p=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)<\sigma}}^{n}\sum_{\substack{k_1,\cdots,k_p=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)<\sigma}}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)$$
$$\leq Q\frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m, g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\frac{\sigma}{2}\right\}\right|+\frac{\sigma}{2}.$$

We determine the sets:

$$H_1 := \left\{(n,m) \in \mathbb{N}^2 : \frac{p!}{(nm)^p} \sum_{j_1,\cdots,j_p=1}^{n} \sum_{k_1,\cdots,k_p=1}^{m} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \geq \sigma\right\},$$

$$H_2 := \left\{(n,m) \in \mathbb{N}^2 : \frac{p!}{(nm)^p} \mid \{j_1, \cdots, j_p \leq n, k_1, \cdots, k_p \leq m, g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \geq \frac{\sigma}{2}\} \mid \geq \frac{\sigma}{2Q}\right\}.$$

If $(m,n) \notin H_2$, then

$$\frac{p!}{(nm)^p}\left|\left\{j_1, \cdots, j_p \leq n, k_1, \cdots, k_p \leq m, g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \geq \frac{\sigma}{2}\right\}\right| < \frac{\sigma}{2Q}$$

In addition, we obtain

Thus $(m,n) \notin H_1$. So, we obtain

$$\left\{(n,m) \in \mathbb{N}^2 : \frac{p!}{(nm)^p} \sum_{j_1,\cdots,j_p=1}^{n} \sum_{k_1,\cdots,k_p=1}^{m} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \geq \sigma\right\}$$
$$\subseteq \left\{(n,m) \in \mathbb{N}^2 : \frac{p!}{(nm)^p} \mid \{j_1, \cdots, j_p \leq n, k_1, \cdots, k_p \leq m, g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \geq \frac{\sigma}{2}\} \mid \geq \frac{\sigma}{2Q}\right\} \in \mathcal{J}_2.$$

As a result, we get $g[C,1,1][\mathcal{J}_2] - \lim_{j,k\to\infty} u_{jk} = u$.

A similar theorem can be established for $gS_{\theta_2}(\mathcal{J}_2)$ and $gN_{\theta_2}[\mathcal{J}_2]$, providing a corresponding relationship between the two.

Theorem 3.1. Assume (Y,g) be a g-metric space and take $(u_{jk}) \in (Y,g)$.

(i) $gN_{\theta_2}[\mathcal{J}_2] - \lim_{j,k\to\infty} u_{jk} = u$ implies $gS_{\theta_2}(\mathcal{J}_2) - \lim_{j,k\to\infty} u_{jk} = u$.

(ii) If g is a bounded function,

$$gS_{\theta_2}(\mathcal{J}_2) - \lim_{j,k\to\infty} u_{jk} = u$$

implies $gN_{\theta_2}[\mathcal{J}_2] - \lim_{j,k\to\infty} u_{jk} = u$.

Theorem 3.2. Consider $\theta = \{(j_r, k_s)\}$ as a lacunary sequence. If $\lim_{\inf_r} q_r > 1$ and $\liminf_s q_s > 1$, then

$$gS_{\mathcal{J}_2} - \lim_{j,k\to\infty} u_{jk} = u \Rightarrow gS_{\theta_2}(\mathcal{J}_2) - \lim_{j,k\to\infty} u_{jk} = u.$$

Proof. Let's assume initially that $\liminf_r q_r > 1$ and $\liminf_s q_s > 1$. In this case, there exist positive constants $\gamma, \tau > 0$ such that $q_r \geq 1+\gamma$ and $q_s \geq 1+\tau$ for sufficiently large values of r and s. This implies that

$$\frac{h_{rs}}{j_r k_s} \geq \frac{\gamma\tau}{(1+\gamma)(1+\tau)}.$$

Assume that $gS_{\mathcal{J}_2} - \lim_{j,k\to\infty} u_{jk} = u$. Then, for all $\sigma > 0$ and for sufficiently large r, s we get

$$\frac{p!}{(j_rk_s)^p} \mid \{j_1 \quad \cdots, j_p \le j_r, k_1, \cdots, k_p \le k_s : g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\} \mid$$

$$\ge \frac{p!}{(j_rk_s)^p}\left|\left\{(j_w, k_w) \in I_{rs}, 1 \le w \le p : g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\}\right|$$

$$= \left(\frac{h_{rs}}{j_rk_s}\right)^p \frac{p!}{(h_{rs})^p}\left|\left\{(j_w, k_w) \in I_{rs}, 1 \le w \le p : g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\}\right|$$

$$\ge \left(\frac{\gamma\tau}{(1+\gamma)(1+\tau)}\right)^p \frac{p!}{(h_{rs})^p}\left|\left\{(j_w, k_w) \in I_{rs}, 1 \le w \le p : g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\}\right|.$$

Then, for any $\delta > 0$, we get

$$\left\{(r,s) \in \mathbb{N}^2 : \frac{p!}{(h_{rs})^p}\left|\left\{(j_w, k_w) \in I_{rs}, 1 \le w \le p : g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\}\right| \ge \delta\right\}$$

$$\subseteq \left\{(r,s) \in \mathbb{N}^2 : \frac{p!}{(j_rk_s)^p} \mid \{j_1, \cdots, j_p \le j_r, k_1, \cdots, k_p \le k_s :\right.$$

$$\left. g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\} \mid \ge \delta\left(\frac{\gamma\tau}{(1+\gamma)(1+\tau)}\right)^p\right\} \in \mathcal{I}_2.$$

As a result, we obtain

$$gS_{\theta_2}(\mathcal{I}_2) - \lim_{j,k\to\infty} u_{jk} = u.$$

Theorem 3.3. If $\theta_2 = \theta_{r,s}$ be a lacunary sequence with $\limsup_r q_r < \infty$, $\limsup_s q_s < \infty$, then

$$gS_{\theta_2}(\mathcal{I}_2) - \lim_{j,k\to\infty} u_{jk} = u \Rightarrow gS_2(\mathcal{I}_2) - \lim_{j,k\to\infty} u_{jk} = u.$$

Proof. Assume $\limsup_r q_r < \infty$, $\limsup_s q_s < \infty$. In this case, there exist positive constants $P, R > 0$ such that $q_r < P$ and $q_s < R$ for all r, s. Considering $gS_{\theta_2}(\mathcal{I}_2) - \lim_{j,k\to\infty} u_{jk} = u$, let's assume that

$$N_{rs} := \left|\left\{(j_w, k_w) \in I_{rs}, 1 \le w \le p : g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\}\right|.$$

Since $gS_{\theta_2}(\mathcal{I}_2) - \lim_{j,k\to\infty} u_{jk} = u$, it supplies for all $\sigma > 0$ and $\delta > 0$,

$$\left\{(r,s) \in \mathbb{N}^2 : \frac{p!}{(h_{rs})^p}\left|\left\{(j_w, k_w) \in I_{rs}, 1 \le w \le p : g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \ge \sigma\right\}\right| \ge \delta\right\}$$

$$\subseteq \left\{(r,s) \in \mathbb{N}^2 : \frac{p!\,N_{rs}}{(h_{rs})^p} \ge \delta\right\} \in \mathcal{I}_2.$$

So, we can select $r_0, s_0 \in \mathbb{N}$ such that $\frac{p!N_{rs}}{(h_{rs})^p} < \delta$ for all $r \ge r_0$ and $s \ge s_0$.

$$Q := \max\{N_{rs} : 1 \le r \le r_0 \text{ and } 1 \le s \le s_0\}$$

and let n, m be integers supplying $j_{r-1} < n \le j_r$ and $k_{s-1} < m \le k_s$. Then, for all $\sigma > 0$,

$$\frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m,g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\right|$$
$$\leq\frac{p!}{(j_{r-1}k_{s-1})^p}\left|\left\{j_1,\cdots,j_p\leq j_r,k_1,\cdots,k_p\leq k_s,g\left(w,w_{i_1},w_{i_2},\dots,w_{i_p}\right)\geq\sigma\right\}\right|$$
$$=\frac{p!}{(j_{r-1}k_{s-1})^p}\left\{\sum_{j_1,\cdots,j_w=1k_1,\cdots,k_w=1}^{r}\sum_{j_1,\cdots,j_w,k_1,\cdots,k_w}^{s}N_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\}$$
$$\leq\frac{p!\,Qr_0^2s_0^2}{(j_{r-1}k_{s-1})^p}+\frac{p!}{(j_{r-1}k_{s-1})^p}\left\{\sum_{j_1,\cdots,j_w=1k_1,\cdots,k_w=1}^{r}\sum_{j_1,\cdots,j_w,k_1,\cdots,k_w}^{s}N_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\}$$
$$\leq\frac{p!\,Qr_0^2s_0^2}{(j_{r-1}k_{s-1})^p}+\frac{p!}{(j_{r-1}k_{s-1})^p}\left\{\sum_{j_1,\cdots,j_w=r_0+1k_1,\cdots,k_w=r_0+1}^{r}\sum_{j_1}^{s}\frac{N_{j_1,\cdots,j_w,k_1,\cdots,k_w}h_{j_1,\cdots,j_w,k_1,\cdots,k_w}}{h_{j_1,\cdots,j_w,k_1,\cdots,k_w}}\right\}$$
$$\leq\frac{p!\,Qr_0^2s_0^2}{(j_{r-1}k_{s-1})^p}+\frac{p!}{(j_{r-1}k_{s-1})^p}\left(\sup_{j_1,\cdots,j_w\geq r_0,k_1,\cdots,k_w\geq r_0}\frac{N_{j_1,\cdots,j_w,k_1,\cdots,k_w}}{h_{j_1,\cdots,j_w,k_1,\cdots,k_w}}\right)\times$$
$$\left\{\sum_{j_1,\cdots,j_w=r_0+1k_1,\cdots,k_w=r_0+1}^{r}\sum_{j_1,\cdots,j_w,k_1,\cdots,k_w}^{s}N_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\}$$
$$\leq\frac{p!\,Qr_0^2s_0^2}{(j_{r-1}k_{s-1})^p}+\delta\left\{\sum_{j_1,\cdots,j_w=r_0+1k_1,\cdots,k_w=r_0+1}^{s}\sum_{j_1,\cdots,j_w,k_1,\cdots,k_w}^{r}N_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\}$$
$$\leq\frac{p!\,Qr_0^2s_0^2}{(j_{r-1}k_{s-1})^p}+\delta PR.$$

Since $j_{r-1}k_{s-1}\to\infty$ as $n,m\to\infty$, it concludes that for each $\sigma>0$

$$\frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m,g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\right|\to 0$$

and as a result for any $\delta>0$, the set

$$\left\{(n,m)\in\mathbb{N}^2:\frac{p!}{(nm)^p}\mid\left\{\left((j_1,\cdots,j_p),(k_1,\cdots,k_p)\right)\in\mathbb{N}^p\times\mathbb{N}^p\right.\right.$$
$$\left.\left.j_1,\cdots,j_p\leq n,k_1,\cdots,k_p\leq m:g\left(u,u_{j_1k_1},\cdots,u_{j_pk_p}\right)\geq\sigma\right\}\mid\geq\delta\right\}\in\mathcal{I}_2.$$

It gives that $gS_2(\mathcal{I}_2)-\lim_{j,k\to\infty}u_{jk}=u$.

In g-metric spaces, we present the relationship between $g[C,1,1][\mathcal{I}_2]$ and $gN_{\theta_2}[\mathcal{I}_2]$.

Theorem 3.4. Considering any lacunary sequence in (Y,g) with $\liminf_r q_r>1$ and $\liminf_s q_s>1$

$$g[C,1,1][\mathcal{I}_2]-\lim_{j,k\to\infty}u_{jk}=u\text{ implies }gN_{\theta_2}[\mathcal{I}_2]-\lim_{j,k\to\infty}u_{jk}=u.$$

Proof. Assume that $\liminf_r q_r>1$ and $\liminf_s q_s>1$. Then, there are $\gamma,\tau>0$ such that $q_r\geq 1+\gamma$ and $q_s\geq 1+\tau$ for sufficiently large r,s which gives that

$$\frac{j_r k_s}{h_{rs}} \leq \frac{(1+\gamma)(1+\tau)}{\gamma\tau} \text{ and } \frac{j_{r-1}k_{s-1}}{h_{rs}} \leq \frac{1}{\gamma\tau}$$

Assume that $g[C,1,1][\mathcal{I}_2] - \lim_{j,k\to\infty} u_{jk} = u$. Then, we write

$$\begin{aligned}
&\frac{p!}{h_{rs}^p} \sum_{(j_w,k_w)\in I_{rs}, 1\leq w\leq p} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \\
&= \frac{p!}{h_{rs}^p} \sum_{j_1,\cdots,j_p=1}^{j_r} \sum_{k_1,\cdots,k_p=1}^{k_s} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \\
&- \frac{p!}{(h_{rs})^p} \sum_{j_1,\cdots,j_p=1}^{j_{r-1}} \sum_{k_1,\cdots,k_p=1}^{k_{s-1}} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \\
&= \frac{(j_rk_s)^p}{(h_{rs})^p}\left[\frac{p!}{(j_rk_s)^p} \sum_{j_1,\cdots,j_p=1 k_1,\cdots,k_p=1}^{j_r} \sum_{k_s} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right)\right] \\
&- \frac{(j_{r-1}k_{s-1})^p}{(h_{rs})^p}\left[\frac{p!}{j_{u-1}k_{s-1}} \sum_{j_1,\cdots,j_p=1}^{j_{r-1}} \sum_{k_1,\cdots,k_p=1}^{k_{s-1}} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right)\right].
\end{aligned}$$

Since $g[C,1,1][\mathcal{I}_2] - \lim_{j,k\to\infty} u_{jk} = u$, then we get

$$\frac{p!}{(j_rk_s)^p} \sum_{j_1,\cdots,j_p=1}^{j_r} \sum_{k_1,\cdots,k_p=1}^{k_s} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \xrightarrow{\mathcal{I}_2} 0 \text{ and}$$
$$\frac{p!}{j_{u-1}k_{s-1}} \sum_{j_1,\cdots,j_p=1}^{j_{r-1}} \sum_{k_1,\cdots,k_p=1}^{k_{s-1}} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \xrightarrow{\mathcal{I}_2} 0.$$

Upon examining the aforementioned equality, we obtain the following relation:

$$\frac{p!}{h_{rs}^p} \sum_{(j_w,k_w)\in I_{rs},} g\left(u, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \xrightarrow{\mathcal{I}_2} 0.$$

Namely, $gN_{\theta_2}[\mathcal{I}_2] - \lim_{j,k\to\infty} u_{jk} = u$.. So, we obtain $g[C,1,1][\mathcal{I}_2] \subseteq gN_{\theta_2}[\mathcal{I}_2]$.

Theorem 3.5. Assume (u_{jk}) be a sequence in g-metric space (Y,g) so that $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u_0$ and $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u_1$, then $u_0 = u_1$.

Proof. Let (u_{jk}) be a double sequence in g-metric space such that $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u_0$ and $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u_1$. For arbitrary $\sigma,\delta > 0$, establish the sets:

$$T_1 = \left\{(n,m)\in\mathbb{N}^2 : \frac{p!}{(nm)^p} \mid \left\{j_1,\cdots,j_p \leq n, k_1,\cdots,k_p \leq m : g\left(u_0, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \geq \sigma\right\} \mid < \delta\right\} \in \mathcal{F}(\mathcal{I}_2)$$

and

$$T_2 = \left\{(n,m)\in\mathbb{N}^2 : \frac{p!}{(nm)^p} \mid \left\{j_1,\cdots,j_p \leq n, k_1,\cdots,k_p \leq m : g\left(u_1, u_{j_1k_1}, \cdots, u_{j_pk_p}\right) \geq \sigma\right\} \mid < \delta\right\} \in \mathcal{F}(\mathcal{I}_2).$$

So, $T_1 \cap T_2 \neq \emptyset$, since $T_1 \cap T_2 \in \mathcal{F}(\mathcal{I}_2)$. Take $(s,q) \in T_1 \cap T_2$ and taket $\sigma = \frac{g(w_0,w_1)}{3} > 0$, so

$$\frac{p!}{(sq)^p}\left|\left\{j_1,\cdots,j_p \le s, k_1,\cdots,k_p \le q: g\left(u_0,u_{j_1k_1},\cdots,u_{j_pk_p}\right) \ge \sigma\right\}\right| < \delta$$

and

$$\frac{p!}{(sq)^p}\left|\left\{j_1,\cdots,j_p \le s, k_1,\cdots,k_p \le q: g\left(u_1,u_{j_1k_1},\cdots,u_{j_pk_p}\right) \ge \sigma\right\}\right| < \delta$$

i.e, for maximum $j_1,\cdots,j_p \le s, k_1,\cdots,k_p \le q$ will supply

$$g\left(u_0,u_{j_1k_1},\cdots,u_{j_pk_p}\right) < \sigma,$$

and

$$g\left(u_1,u_{j_1k_1},\cdots,u_{j_pk_p}\right) < \sigma$$

for a small $\delta > 0$. As a result, we have to get

$$\left\{j_1,\cdots,j_p \le s, k_1,\cdots,k_p \le q, g\left(u_0,u_{j_1k_1},\cdots,u_{j_pk_p}\right) < \sigma\right\} \cap \left\{j_1,\cdots,j_p \le s, k_1,\cdots,k_p \le q: g\left(u_1,u_{j_1k_1},\cdots,u_{j_pk_p}\right) < \sigma\right\} \neq \emptyset$$

a contradiction, as the neighborhood of u_0 and u_1 are disjoint. So, $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk}$ determined uniquely.

Theorem 3.6. For any sequence $(u_{jk}) \in (Y,g)$, $gS_2 - \lim_{j,k\to\infty} u_{jk} = u_0$ implies $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u_0$.

Proof. Suppose $gS_2 - \lim_{j,k\to\infty} u_{jk} = u_0$. Then, for all $\sigma > 0$, the set

$$A(\sigma) = \left\{j_1,\cdots,j_p \le n, k_1,\cdots,k_p \le m: g\left(u_0,u_{j_1k_1},\cdots,u_{j_pk_p}\right) \ge \sigma\right\}$$

has natural density zero. So, we get

$$\lim_{n,m\to\infty} \frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p \le n, k_1,\cdots,k_p \le m: g\left(u_0,u_{j_1k_1},\cdots,u_{j_pk_p}\right) \ge \sigma\right\}\right| = 0.$$

As a result, for all $\sigma, \delta > 0$,

$$\left\{(n,m) \in \mathbb{N}^2: \frac{p!}{(nm)^p}\left|\left\{j_1,\cdots,j_p \le n, k_1,\cdots,k_p \le m: g\left(u_0,u_{j_1k_1},\cdots,u_{j_pk_p}\right) \ge \sigma\right\}\right| \ge \delta\right\}$$

is a finite set and so belongs to $\mathcal{I}_2$, where $\mathcal{I}_2$ is an admissible ideal. Therefore, we obtain $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u_0$.

Theorem 3.7. For any sequence (u_{jk}), $g\mathcal{I}_2 - \lim_{j,k\to\infty} u_{jk} = u_0$, means $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u_0$.

Proof. The proof is obvious. But the vice versa is not supplying.

Example 3.1. Let $X = \mathbb{R}$ and g be the metric as follows;

$$\begin{aligned} g &: \mathbb{R}^3 \to \mathbb{R}^+ \\ g(u,v,w) &= \max\{|u-v|,|u-w|,|v-w|\}. \end{aligned}$$

When we assume $\mathcal{I}_2 = \mathcal{I}_2^f$ the sequence (u_{jk}), where

$$u_{jk} = \begin{cases} 0, & \text{if } j = u^2, k = v^2, u, v \in \mathbb{N} \\ 1, & \text{otherwise} \end{cases}$$

Then, $S_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = 1$. But (u_{jk}) is not $g\mathcal{I}_2$-convergent.

Theorem 3.8. If all subsequence of (u_{jk}) is $g\mathcal{I}_2$-statistically convergent to u, then $gS_{\mathcal{I}_2} - \lim_{j,k\to\infty} u_{jk} = u$

Proof. The proof is clear, so omitted.

CONCLUSION

In this article, we have extensively explored the concepts of $\mathcal{I}_2$-statistical convergence and $\mathcal{I}_2$-lacunary statistical convergence for sequences in general metric spaces, with a specific focus on g-metric spaces. By delving into the intricacies of these notions, we have gained a deeper understanding of the behavior of sequences within the framework of g-metric spaces.

Throughout our investigation, we have examined the properties and characteristics of $\mathcal{I}_2$-statistical convergence and $\mathcal{I}_2$-lacunary statistical convergence, shedding light on their significance in the context of general metric spaces. The study has allowed us to establish a clear relationship between these convergence concepts and the underlying g-metric structures, enabling a more comprehensive analysis of sequence convergence.

By expanding our understanding of these convergence notions in g-metric spaces, we have contributed to the existing body of knowledge in the field. The insights gained from our exploration can pave the way for further research and applications in various areas that utilize g-metric spaces, such as analysis, functional analysis, and other related fields.

In summary, our investigation into $\mathcal{I}_2$-statistical convergence and $\mathcal{I}_2$-lacunary statistical convergence in g-metric spaces has provided valuable insights and expanded our understanding of these convergence concepts. We anticipate that this research will inspire future studies and contribute to the advancement of mathematical analysis in general metric spaces.

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