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Further Results on Lacunary Difference Sequences of Complex Uncertain Variables in 2-Normed Spaces

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Abstract –A novel approach combining the difference operator on sequence spaces and uncertainty theory has been utilized to establish a fresh class of lacunary convergent difference sequences involving complex uncertain variables in 2-normed spaces. This newly introduced class demonstrates remarkable properties related to lacunary convergence. Additionally, the sequence spaces defined in this context have been thoroughly explored, revealing interesting topological characteristics and valuable insights into inclusion relations. This study offers a rejuvenated perspective on lacunary convergence and its associated concepts through the integration of difference operators and uncertainty theory.

Keywords – Uncertainty Theory, Complex Uncertain Variable, 2-Normed Space, Difference Sequence

I. INTRODUCTION

The notion of uncertainty theory was originally proposed by Liu [1] in 2007. Over time, uncertainty theory has gained significant importance as a focal point of research in various branches of mathematics, including uncertain programming, uncertain risk analysis, uncertain logic, and more.

In our daily lives, we encounter different forms of uncertainty, such as randomness and fuzziness. Probability theory is commonly employed to model the frequencies of random events, while uncertainty theory focuses on quantifying the degree of belief in the truth of an event. The foundation of uncertainty theory lies in uncertain measures, which adhere to axioms such as normality, duality, subadditivity, and product axioms.

Complex uncertain variables, on the other hand, are measurable functions that map uncertainty spaces to the set of complex numbers. The convergence of sequences assumes a vital role across various mathematical theories. Chen et al. [2] initially introduced the concept of convergence for complex uncertain variables. Subsequent investigations into complex uncertain variables have been conducted by Tripathy and Nath [3], Dowari and Tripathy [4,5] and numerous other researchers.

The foundational research on lacunary sequences was conducted by Freedman et al. [6]. They investigated the behavior of the modulus of the strongly Cesàro summable sequences, specifically focusing on sequences with a general lacunary parameter θ. As a result, they introduced a broader class of sequences known as lacunary sequences, denoted by N_θ. Since then, their findings have served as a basis for further exploration by numerous researchers.

The concept of difference sequence was originally presented by Kizmaz [7]. Subsequently, Esi et al. [8] extended this notion by introducing the generalized difference sequence spaces. Et and Çolak [9] researched some properties of difference sequences.

The notion of 2-normed spaces was originally introduced by Gähler in [10]. Since its inception, this concept has been extensively explored by numerous researchers, leading to a wealth of results and findings, as documented in [11].

In this study, we build upon Gähler's work by introducing the concept of statistical convergence in 2-normed spaces. The primary objective of this current study is to extend these concepts to include uncertain sequences within the framework of uncertainty space. Specifically, we aim to explore the lacunary convergence concepts of complex uncertain sequences concerning the difference sequence in 2-normed spaces.

II. Definitions and Basic Properties

The purpose of this section is to bring together the essential data and methods needed to achieve our main objectives. We will start by introducing several crucial terms.

In their research, Freedman et al. [6] conducted a comprehensive study on a particular class of sequences known as lacunary strongly convergent sequences, denoted as N_θ. The definition of N_θ, as provided by the authors, is as follows:

$$N_\theta = \left\{ x = (x_k): \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0, \text{ for some } L \right\}$$

Kizmaz [7] introduced the concept of a difference sequence, which laid the foundation for further developments in the field. Building upon this work, Esi, Tripathy, and Sarma [8] extended the concept and introduced the notion of generalized difference sequence spaces, defining them in the following manner:

Consider fixed integers m and n, where both m and n are greater than or equal to zero.

$\mathrm{Z}(\triangle_m^n) = \{x = (x_k) \in \omega : \triangle_m^n x = (\triangle_m^n x_k) \in \mathrm{Z}\}$

for $\mathrm{Z} = \ell_\infty, \mathrm{c}$ and c_0; where $\triangle_m^n x_k = (\triangle_m^{n-1} x_k - \triangle_m^{n-1} x_{k+m})$ and $\triangle_m^0 x_k = x_k$ for each $k \in \mathbb{N}$.

The generalized difference notion can be represented using the following binomial expression:

$$\triangle_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv} \quad \text{for all } k \in \mathbb{N}$$

For $m = 1$ and $n = 1$, these spaces represent the spaces $\ell_\infty(\triangle), \mathrm{c}(\triangle)$ and $\mathrm{c}_0(\triangle)$ put forward and worked by Kizmaz [7]. When considering the case where m is equal to 1, the spaces obtained represent the well-known spaces $\ell_\infty(\triangle^n), \mathrm{c}(\triangle^n)$ and $\mathrm{c}_0(\triangle^n)$ that were investigated and extensively studied by Et and Çolak [9]. For $n = 1$, these spaces represent the spaces $\ell_\infty(\triangle_m), \mathrm{c}(\triangle_m)$ and $\mathrm{c}_0(\triangle_m)$ introduced and studied by Tripathy and Esi [12].

The sequence spaces $\mathrm{Z}(\triangle_m^n)$ for $\mathrm{Z} = \ell_\infty, \mathrm{c}$ and c_0 are Banach spaces, by the norm

$$\| x \|_{\triangle_m^n} = \sum_{i=1}^{p} |x_i| + \sup_k |\triangle_m^n x_k|$$

where $p = mn$ for $m \geq 1, n \geq 1$.

Let X be a real vector space of dimension d; where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\|: X \times X \to \mathbb{R}$ which supplies i) $\|x,y\| = 0$ iff x and y are linearly dependent; ii) $\|x,y\| = \|y,x\|$; iii) $\|\alpha x, y\| = |\alpha| \|x,y\|$, $\alpha \in \mathbb{R}$; iv) $\|x, y+z\| \leq \|x,y\| + \|x,z\|$. The pair $(X, \|.,.\|)$ is named a 2-normed space.

Let us now provide a brief overview of the uncertainty theory, which was introduced in reference [1].

Consider a nonempty set Γ with a σ-algebra $\mathcal{L}$ defined on it. An uncertain measure is a set function $\mathcal{M}$ that satisfies the following axioms:

A1. $\mathcal{M}\{\Gamma\} = 1$;

A2. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;

A3. For each countable sequence of $\{\lambda_j\} \in \mathcal{L}$, we get

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\lambda_j\}.$$

An uncertainty space is defined as a triplet $(\Gamma, \mathcal{L}, \mathcal{M})$, where Γ is a nonempty set, $\mathcal{L}$ is a σ-algebra on Γ, and $\mathcal{M}$ is an uncertain measure. To determine the uncertain measure of a compound event, Liu [1] introduced a concept called a product uncertain measure, defined as follows:

A4. Consider uncertainty spaces $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ where each k corresponds to a specific uncertainty space. The product uncertain measure, denoted as $\mathcal{M}$, is a measure that satisfies the following properties:

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k represents arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1,2, \ldots$, respectively.

An uncertain variable is a measurable function ξ that maps an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers. In other words, for any Borel set B of complex numbers, the set

$$\{\xi \in \mathcal{B}\} = \{\gamma \in \Gamma : \xi(\gamma) \in \mathcal{B}\}$$

is considered an event. When the range of ξ is limited to the set of real numbers, it is referred to as an uncertain variable, which was introduced and studied by Liu [1]. Complex uncertain variables, as complex functions defined on uncertainty spaces, are primarily employed to model complex uncertain quantities.

The expected value operator for an uncertain variable was defined by Liu [1] as follows:

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr.$$

This definition holds provided that at least one of the two integrals is finite. The expected value operator serves as a measure to quantify the average value of an uncertain variable.

The complex uncertainty distribution $\Phi(x)$ of a complex uncertain variable ξ is defined as a function from the complex plane $\mathbb{C}$ to the interval $[0,1]$. It is given by:

$$\Phi(c) = \mathcal{M}\{\mathrm{Re}\,(\xi) \leq \mathrm{Re}\,(c), \mathrm{Im}\,(\xi) \leq \mathrm{Im}\,(c)\}$$

where c is any complex number. The value of $\Phi(c)$ represents the probability that both the real part of ξ and the imaginary part of ξ are less than or equal to the corresponding real and imaginary parts of c, respectively.

An uncertain variable is classified as positive when its domain is the set of non-negative real numbers, including zero, and it ranges between 0 and 1. Recognizing the significant role of sequence convergence in mathematics, Chen, Ning, and Wang [2] introduced various concepts of convergence for complex uncertain sequences. Complex uncertain sequences are sequences of complex uncertain variables indexed by integers.

The complex uncertain sequence $\{\xi_n\}$ is considered to be convergent almost surely (a.s.) to L if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} \|\xi_n(\gamma) - L(\gamma)\| = 0,$$

holds for each $\gamma \in \Lambda$. In this situation, we denote the convergence as$\xi_n \to \mathrm{L}$, a.s.

The complex uncertain sequence $\{\xi_n\}$ is considered to be convergent in measure to L provided that for a given $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathcal{M}\{\|\xi_n(\gamma) - \mathrm{L}(\gamma)\| \geq \varepsilon\} = 0$$

The complex uncertain sequence $\{\xi_n\}$ is defined as convergent in mean to L if

$$\lim_{n\to\infty} \mathrm{E}[\|\xi_n(\gamma) - \mathrm{L}(\gamma)\|] = 0$$

Let $\Phi, \Phi_1, \Phi_2, \Phi_3, \ldots$ be the complex uncertainty distributions of complex uncertain variables $\xi, \xi_1, \xi_2, \xi_3 \ldots$, respectively. We say the complex uncertain sequence $\{\xi_n\}$ converges in distribution to L if

$$\lim_{n\to\infty} \Phi_n(c) = \Phi(c)$$

for all $c \in \mathbb{C}$, at which $\Phi(c)$ is continuous.

The complex uncertain sequence $\{\xi_n\}$ is said to be convergent uniformly almost surely (u.a.s.) to L if there exists a sequence of events $\{E'_k\}, \mathcal{M}\{E'_k\} \to 0$ such that $\{\xi_n\}$ converges uniformly to L in $\Gamma - E'_k$, for any fixed $k \in \mathbb{N}$.

III. MAIN RESULTS

In this section, we introduce novel categories of sequences composed of uncertain variables by employing the difference operator on sequences in 2-normed spaces. By doing so, we redefine and invigorate the understanding of these classes of sequences.

$$\left\|\sigma_1, \|.,.\|\right\|^U (\Delta_m^n) = \left\{ w = (w_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|\triangle_m^n w_k(\gamma), z(\gamma)\| = 0 \right\}.$$

Assume that θ be a lacunary sequence and Consider a complex uncertain sequence denoted as $\{w_n\}$ within 2-normed spaces $(X, \|.,.\|)$.

Before we state the main results of this work, let us give the definition of a new statistical method.

We define the following classes of sequences:

$$\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_0=\left\{w=(w_k):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma),\mathrm{z}(\gamma)\|=0\right\},$$

$$\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_1=\left\{w=(w_k):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma)-w(\gamma),\mathrm{z}(\gamma)\|=0\right.$$

for some $w(\gamma)\in(\Gamma,\mathcal{L},\mathcal{M})\}$,

$$\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_\infty=\left\{w=(w_k):\sup_r\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma),\mathrm{z}(\gamma)\|<\infty\right\},$$

$$[|AC_\theta|^U,\triangle_m^n,\|.,.\|]=\left\{w=(w_k):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_{k+n}(\gamma)-w(\gamma)\|,\mathrm{z}(\gamma)=0,\text{ uniformly in } n\right\}.$$

A sequence space E is considered solid (or normal) if for any sequence (x_k) belonging to E and for all sequences (α_k) of scalars satisfying $k\in\mathbb{N}$, the sequence $(\alpha_k x_k)$ also belongs to E. On the other hand, a sequence space is said to be symmetric if whenever (x_k) belongs to E, its permuted sequence $(x_{\pi(k)})$ also belongs to E, where π represents a permutation of $\mathbb{N}$. Lastly, a sequence space E is considered monotone if it contains all the canonical pre-images of its step spaces.

Lemma 3.1. The implication holds that if a sequence space is solid, it is also monotone.

Now, with the newly introduced concepts in mind, we proceed to explore the following results.

Theorem 3.1. The classes of complex uncertain sequences $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_0$, $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_1$ and $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_\infty$ are linear 2-normed spaces.

Proof. We present the outcome for the class of complex uncertain sequences $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_0$. Similar arguments can be applied to other cases as well. Consider (w_k) and (t_k) are belonging to $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_0$. Then we have,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma),\mathrm{z}(\gamma)\|=0,$$

$$\text{and }\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n t_k(\gamma),\mathrm{z}(\gamma)\|=0.$$

Now for $\alpha,\beta\in\mathbb{C}$ and for each $z\neq 0$ in $(X,\|.,.\|)$,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n(\alpha w_k(\gamma)+\beta t_k(\gamma)),\mathrm{z}(\gamma)\|$$
$$=\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\alpha\triangle_m^n w_k(\gamma)+\beta\triangle_m^n t_k(\gamma),\mathrm{z}(\gamma)\|$$
$$\leq|\alpha|\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma),\mathrm{z}(\gamma)\|$$
$$+|\beta|\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n t_k(\gamma),\mathrm{z}(\gamma)\|\to 0,\text{ as } r\to\infty.$$

So $(\alpha w_k(\gamma)+\beta t_k(\gamma))\in\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_0$.

As a result, $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_0$ is a linear 2-normed space.

Theorem 3.2. The classes of complex uncertain sequences $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_0$, $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_1$ and $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]_\infty$ are 2-normed linear spaces, normed by

$$\|w(\gamma),\mathrm{z}(\gamma)\|_{\triangle_m^n}=\sum_{i=1}^{p}\|w_i(\gamma),\mathrm{z}(\gamma)\|+\sup_r\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma),\mathrm{z}(\gamma)\|.$$

where $p=mn$ for $m\geq 1, n\geq 1$, for each $z\neq 0$ in $(X,\|.,.\|)$.

Proof. For $w=\theta$, we get $\|\theta\|_{\Delta_m^n}=0$.

Conversely, consider the following

$$\| w(\gamma), z(\gamma) \|_{\Delta_m^n} = 0.$$

So

$$\| w(\gamma), z(\gamma) \|_{\triangle_m^n} = \sum_{i=1}^{p} \|w_i(\gamma), z(\gamma)\| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n w_k(\gamma), z(\gamma)\| = 0,$$

for each $z \neq 0$ in $(X, \|.,.\|)$. So, $w_i(\gamma) = 0$, for $i = 1,2,\dots,mn$ and $\|\triangle_m^n w_k(\gamma), z(\gamma)\| = 0$ for $k \in I_r, r = 1,2,\dots$ and for each $z \neq 0$ in $(X, \|.,.\|)$.

Let $k = 1$, i.e.,

$\|\triangle_m^n w_1(\gamma), z(\gamma)\| = 0 \Rightarrow \|\triangle_m^{n-1} w_1(\gamma) - \triangle_m^{n-1} w_{m+1}(\gamma), z(\gamma)\| = 0 \Rightarrow w_{m+1}(\gamma) = 0$, since $w_i(\gamma) = 0$, for $i = 1,2,\dots,mn$.

By applying the principle of induction, we can establish that $w_k(\gamma) = 0$, for all $k \in I_r$.

$$\| w(\gamma) + t(\gamma), z(\gamma) \|_{\Delta_m^n} = \sum_{i=1}^{p} \|w_i(\gamma) + t_i(\gamma), z(\gamma)\| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n \big(w_k(\gamma) + t_k(\gamma)\big), z(\gamma)\| \le \sum_{i=1}^{p} \|w_i(\gamma), z(\gamma)\| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n w_k(\gamma), z(\gamma)\| + \sum_{i=1}^{p} \|t_i(\gamma), z(\gamma)\| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \|\Delta_m^n t_k(\gamma), z(\gamma)\| = \| w(\gamma), z(\gamma) \|_{\Delta_m^n} + \| t(\gamma), z(\gamma) \|_{\Delta_m^n},$$

for each $z \neq 0$.

For any $\lambda \in \mathbb{C}$,

$$\| \lambda w(\gamma), z(\gamma) \|_{\Delta_m^n} = \sum_{i=1}^{p} \|\lambda w_i(\gamma), z(\gamma)\| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n \big(\lambda w_k(\gamma)\big), z(\gamma)\| = |\lambda| \| w(\gamma), z(\gamma) \|_{\Delta_m^n}.$$

This finalizes the proof.

Theorem 3.3. For $m \geq 1$ and $n \geq 1$

$$\left[N_\theta^U, \triangle_m^{n-1}, \|.,.\|\right]_q \subset \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_q$$

for $q = 0,1,\infty$.

Generally, $\left[N_\theta^U, \triangle_m^i, \|.,.\|\right]_q \subset \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_q$ for $q = 0,1,\infty$ and $i = 0,1,\dots,n-1$. The inclusions are strict.

Proof. Let $\{w_k\} \in \left[N_\theta^U, \triangle_m^{n-1}, \|.,.\|\right]_0$.

Then, we get

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^{n-1} w_k(\gamma), z(\gamma)\| = 0. \quad (3.1)$$

Now,

$$\frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n w_k(\gamma), z(\gamma)\| = \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^{n-1} w_k(\gamma) - \triangle_m^{n-1} w_{k+1}(\gamma), z(\gamma)\| \le \left(\frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^{n-1} w_k(\gamma), z(\gamma)\| - \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^{n-1} w_{k+1}(\gamma), z(\gamma)\| \right)$$

As we approach the limit as r tends to infinity, we obtain the following result:

$$\frac{1}{h_r}\sum_{k\in I_r} \|\triangle_m^n w_k(\gamma), z(\gamma)\| = 0, \text{ by } (3.1)$$

which gives $\{w_k\} \in \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$.

Similarly, the remaining cases can be established using the same approach. By proceeding inductively, we can derive the following: $\left[N_\theta^U, \triangle_m^i, \|.,.\|\right]_q \subset \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_q \infty$ and $i = 0,1,\dots,n-1$.

The above inclusion is strictly valid. To illustrate this, consider the following example that highlights its strictness:

Example 3.1. Let us consider the lacunary sequence $\theta = (2^r)$ and the sequence of uncertain variables to be $(w_k) = (k_{n-1})$. Then $\triangle_m^n (w_k) = 0, \triangle_m^n x_k = \sum_{v=0}^{n-1}(-1)^v \binom{n-1}{v} x_{k+mv}$, for all $k \in \mathbb{N}$. As a result $(w_k) \in \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$ but $(w_k) \notin \left[N_\theta^U, \triangle_m^{n-1}, \|.,.\|\right]_0$.

Theorem 3.4. The spaces of uncertain sequences $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_Z$, $Z = 0,1,\infty$ do not exhibit monotonicity.

To demonstrate the lack of monotonicity in the spaces, let us consider the following example.

Example 3.2. To illustrate the lack of monotonicity in $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$, we will focus on the case where $m = n = 2$. Let us consider the lacunary sequence $\theta = (2^r)$ and the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_k\in\Lambda} \dfrac{k}{2k+1}, & \text{if } \sup\limits_{\gamma_k\in\Lambda} \dfrac{k}{2k+1} < 0.5 \\ 1 - \sup\limits_{\gamma_k\in\Lambda^c} \dfrac{k}{2k+1}, & \text{if } \sup\limits_{\gamma_k\in\Lambda^c} \dfrac{k}{2k+1} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Consider uncertain variables as

$$w_k(\gamma_j) = \begin{cases} k, & \text{if } j = k \\ 0, & \text{if not.} \end{cases}$$

Then, it can be verified that the sequence $\{w_k\}$ for $k \in I_r$ and $r = 1,2,3,\dots$ is in $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$. Consider the sequence $\{t_i\}$ which is a rearrangement of the sequence $\{w_k\}$ given by $t_i(\gamma) =$ defined as $\{\gamma_1, \gamma_2, \dots\}$ with the power set. For any event $\Lambda \in \mathcal{L}$, we have

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_k\in\Lambda} \frac{1}{2^k}$$

Let us define the uncertain variables as follows:

$$w_k(\gamma_j) = \begin{cases} 2^k, \text{ if } j = k, \\ 0, \text{ otherwise.} \end{cases}$$

Then $\{w_k\} \in \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$. Consider the pre-image space of the sequence $\{t_k\}$ defined by

$$t_k(\gamma_j) = \begin{cases} w_k, & \text{if } k = i^2 \in \mathbb{N}, \\ 0, \text{ otherwise.} \end{cases}$$

Then $\{t_k\} \notin \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$. Therefore, the space is not monotone. The fact that the spaces are not solid can be deduced from Lemma 3.1.

Theorem 3.5. The spaces of uncertain sequences $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_Z, Z = 0,1,\infty$ are not symmetric.

For this we consider the following example.

Example 3.3. Let us demonstrate this for the case of $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$. It is worth noting that similar examples can be constructed for the other two spaces.

For $m = n = 2$, we consider the lacunary sequence $\theta = (2^r)$. Furthermore, let the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ be defined as $\{\gamma_1, \gamma_2, \gamma_3, \dots\}$ with the power set. For any event $\Lambda \in \mathcal{L}$ so that

$\{w_1, w_4, w_9, w_2, w_{10}, \dots\} \notin \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$. Thus $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_0$ are not symmetric in general.

Next we prove some inclusion results, for our convenience we shall denote $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]_Z$ for $Z = 0,1,\infty$ by $\left[N_\theta^U, \triangle_m^n, \|.,.\|\right]$.

Theorem 3.6. $|\sigma_1, \|.,.\||^U(\triangle_m^n) \subset \left[N_\theta^U, \triangle_m^n, \|.,.\|\right]$ iff $\liminf_r q_r > 1$.

Proof. Let $\lim \inf q_r > 1$, then there exists $\delta > 0$ such that, $1 + \delta \le q_r$ for all $r \ge 1$.

For $w = \{w_i(\gamma)\} \in |\sigma_1, \|.,.\||^U(\triangle_m^n)$ we obtain $\tau_r = \frac{1}{h_r}\sum_{i=1}^{k_r} \|\triangle_m^n w_i(\gamma), z(\gamma)\| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} \|\triangle_m^n w_i(\gamma), z(\gamma)\| = \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r} \|\triangle_m^n w_i(\gamma), z(\gamma)\|\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_r} \|\triangle_m^n w_i(\gamma), z(\gamma)\|\right)$.

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$.

Now, $\frac{1}{k_r}\sum_{i=1}^{k_r}\|\triangle_m^n w_i(\gamma), z(\gamma)\|$ and $\frac{1}{k_{r-1}}\sum_{i=1}^{k_r}\|\triangle_m^n w_i(\gamma), z(\gamma)\|$ both converges to 0, namely, $\{w_i(\gamma)\} \in [N_\theta^U, \triangle_m^n, \|.,.\|]$. Therefore, $|\sigma_1, \|.,.\||^U(\triangle_m^n) \subset [N_\theta^U, \triangle_m^n, \|.,.\|]$.

Conversely suppose that $\liminf_r q_r = 1$. Since θ is lacunary, we can obtain a subsequence $\{k_{r_j}\}$ of θ satisfying, $\frac{k_{r_j}^r}{k_{r_j-1}} < 1 + \frac{1}{j}$ and $\frac{k_{r_j-1}}{k_{r_{j-1}}} > j$, where $r_j \geq r_{j-1} + 2$. Assume that $\zeta(\gamma)$ and $\eta(\gamma)$ be two distinct uncertain variables. Establish $w = \{w_k(\gamma)\}$ as

$$\triangle_m^n w_i(\gamma) = \begin{cases} \zeta, & \text{if } i \in I_{r_j} \text{ for some } j \in \mathbb{N} \\ \eta, & \text{if not.} \end{cases}$$

For any uncertain variable $\rho(\gamma)$, we obtain

$$\frac{1}{h_{r_j}}\sum_{I_{r_j}} \|\triangle_m^n w_i(\gamma) - \zeta(\gamma), z(\gamma)\| = \|\zeta(\gamma) - \rho(\gamma)\|; j = 1,2, \dots$$

and

$$\frac{1}{h_r}\sum_{I_r} \|\triangle_m^n w_i(\gamma) - \eta(\gamma)\| = \|\eta(\gamma) - \rho(\gamma)\|; \text{ for } r \neq r_j.$$

It follows that $\{w_i(\gamma)\} \notin [N_\theta^U, \triangle_m^n, \|.,.\|]$. Assuming that t is sufficiently large, there exists a unique index j that satisfies the condition: $k_{r_j-1} < t \leq k_{r_{j+1}-1}$.

$$\frac{1}{t}\sum_{i=1}^{t} \|\triangle_m^n w_i(\gamma), z(\gamma)\| \leq \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j} - 1} \leq \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

When $t \to \infty$, it gives that $j \to \infty$. So $\{w_i(\gamma)\} \in |\sigma_1, \|.,.\||^U(\triangle_m^n)$. Thus $\{w_i(\gamma)\}$ is strongly summable.

Theorem 3.7. $[N_\theta^U, \triangle_m^n, \|.,.\|] \subset |\sigma_1, \|.,.\||^U(\triangle_m^n)$ iff $\limsup_r < \infty$.

Proof. Let limsup $q_r < \infty$, there exists $H > 0$ such that $q_r < H$ for all $r \geq 1$. Considering $\xi = \{\xi_i(\gamma)\} \in [N_\theta^U, \triangle_m^n]$ and $\varepsilon > 0$ we can find $R > 0$ and $K > 0$ such that $\sup\tau_i < \varepsilon, \tau_i < K$ for all $i = 1,2, \dots$ Then if t is any integer with $k_{r-1} < t \leq k_r$, where $i \geq R$

$r > R$, and that t is any integer with $k_{r-1} < t \leq k_r$, then we can write

$$\begin{aligned}
&\frac{1}{t}\sum_{i=1}^{t} \|\triangle_m^n w_i(\gamma), z(\gamma)\| \\
&\leq \frac{1}{k_{r-1}}\sum_{i=1}^{t} \|\triangle_m^n w_i(\gamma), z(\gamma)\| \\
&= \frac{1}{k_{r-1}}\left(\sum_{I_1} \|\triangle_m^n w_i(\gamma), z(\gamma)\| + \dots + \sum_{I_r} \|\triangle_m^n w_i(\gamma), z(\gamma)\|\right) \\
&= \frac{1}{k_{r-1}}\tau_1 + \frac{k_2 - k_1}{k_{r-1}}\tau_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}}\tau_r + \frac{k_{R+1} - k_R}{k_{r-1}}\tau_{R+1} + \dots + \frac{k_r - k_{r-1}}{k_{r-1}}\tau_r \\
&\leq \left(\sup_{i\geq 1}\tau_i\right)\frac{k_R}{k_{r-1}} + \left(\sup_{i\geq R}\tau_i\right)\frac{k_r - k_R}{k_{r-1}} \\
&= K \cdot \frac{k_R}{k_{r-1}} + \varepsilon.H.
\end{aligned}$$

Since $k_{r-1} \to \infty$ as $t \to \infty$, it follows that $\frac{1}{t}\sum_{i=1}^{t}\|\triangle_m^n w_i(\gamma), z(\gamma)\| \to 0$. i.e., $= \{w_i(\gamma)\} \in |\sigma_1, \|.,.\||^U(\triangle_m^n)$. Suppose $\limsup q_r = \infty$. In order to prove the result we need to find a sequence $w=\{w_i(\gamma)\}$ of uncertain variables such that $w\in\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]$ and $w\notin|\sigma_1,\|.,.\||^U(\triangle_m^n)$. We select a subsequence k_{r_j} of lacunary θ such that $q_{r_j}>j$. Let $\zeta(\gamma)$ and $\eta(\gamma)$ be two distinct uncertain variables and then define $w=\{w_i(\gamma)\}$ by

$$w_i(\gamma)=\begin{cases}\zeta, & \text{if } k_{r_{j-1}}<i\le 2k_{r_{j-1}}, \text{ for some } j\in\mathbb{N}\\ \eta, & \text{otherwise}\end{cases}$$

Then

$$\tau_{r_j}=\frac{1}{h_{r_j}}\sum_{I_{r_j}}\|\triangle_m^n\,\xi_i(\gamma)-\eta(\gamma),\mathrm{z}(\gamma)\|=\|\zeta(\gamma)-\eta(\gamma),\mathrm{z}(\gamma)\|<\frac{k_{r_{j-1}}}{k_{r_j}-k_{r_{j-1}}}<\frac{1}{j-1}$$

and $\tau_r=0$ if $r\neq r_j$. Hence, $\lim_r\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n\,\xi_i(\gamma),\mathrm{z}(\gamma)\|=0$. Thus $\{\xi_i(\gamma)\}\in\left[N_\theta^U,\triangle_m^n\right]$.

Now for the sequence $\{\xi_i\}$ above and for an uncertain variable ρ,

$$\frac{1}{k_{r_j}}\sum_{i=1}^{k_{r_j}}\|\triangle_m^n\,w_i(\gamma)-\rho(\gamma,\mathrm{z}(\gamma))\|\ge\frac{1}{k_{r_j}}\left(\sum_{i=k_{r_{j-1}}}^{2k_{r_j}-1}\|\zeta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|+\sum_{i=2k_{r_{j-1}}}^{2k_{r_j}}\|\eta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\right)\ge$$
$$\|\zeta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\cdot\frac{k_{r_{j-1}}}{k_{r_j}}+\|\eta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\cdot\frac{k_{r_j}-2k_{r_{j-1}}}{k_{r_j}}$$
$$\ge\|\zeta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\cdot\frac{k_{r_{j-1}}}{k_{r_j}}+\|\eta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\cdot\left(1-\frac{2}{j}\right)\to\|\eta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|,$$

and

$$\frac{1}{2k_{r_j}-1}\sum_{i=1}^{2k_{r_j}-1}\|\triangle_m^n\,w_i(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\ge\frac{k_{r_{j-1}}}{2k_{r_{j-1}}}\|\eta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\to\frac{\|\eta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|}{2}.$$

Consequently for any uncertain variable ρ, we have

$$\lim_{j\to\infty}\frac{1}{k_{r_j}}\sum_{i=1}^{k_{r_j}}\|\triangle_m^n\,w_i(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|=\|\zeta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|\neq\frac{\|\eta(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|}{2}=\lim_{j\to\infty}\frac{1}{2k_{r_{j-1}}}\sum_{i=1}^{2k_{r_{j-1}}}\|\triangle_m^n\,w_i(\gamma)-\rho(\gamma),\mathrm{z}(\gamma)\|.$$

Hence $\{w_i(\gamma)\}\notin|\sigma_1,\|.,.\||^U(\triangle_m^n)$.

The following result is the consequence of the above two theorems.

Theorem 3.8. $\left[N_\theta^U,\triangle_m^n,\|.,.\|\right]=|\sigma_1,\|.,.\||^U(\triangle_m^n)$ if and only if $1<\liminf_r q_r<\infty$.

Theorem 3.9.

$$|AC,\|.,.\||^U(\triangle_m^n)\subset\left[N_\theta^U,\triangle_m^n,\|.,.\|\right].$$

Proof. Let $\{w_i(\gamma)\}\in|AC,\|.,.\||^U(\triangle_m^n)$ and $\varepsilon>0$, there exists $N>0$ and $w(\gamma)$ such that

$$\frac{1}{h_r}\sum_{i\in I_r}\|\triangle_m^n\,w_{i+n}(\gamma)-w(\gamma),\mathrm{z}(\gamma)\|<\varepsilon\ \text{ for }\ n>N,\ r=1,2,3,\ldots$$

Given that θ is a lacunary sequence, we can select a positive value R such that for $r\ge R$, h_r exceeds N, resulting in τ_r being less than ε. As a consequence, the sequence $\{w_i(\gamma)\}$ belongs to the set N_θ. To construct a sequence that lies in N_θ but not in $|AC|$, we define w as follows:

$$w_i(\gamma) = \begin{cases} 1, & \text{if for some } r,\ k_{r-1} < i \le k_{r-1} + \sqrt{h_r} \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, w consists of sequences of 0's and 1's that can be arbitrarily long. This observation leads to the conclusion that w is not strongly almost convergent.

But, $\tau_r = \frac{1}{h_r}\sum_{I_r} \|w_i(\gamma), z(\gamma)\| = \frac{1}{h_r}\left[\sqrt{h_r}\right] = \frac{1}{\sqrt{h_r}}$ which converges to 0 as $r \to \infty$.

In this section, we introduce and define the concepts of lacunary convergence for difference uncertain sequences in 2-normed spaces. Furthermore, we establish the relationships between these different forms of convergence.

Let $\{w_k\}$ be a complex uncertain sequence in 2-normed spaces $(X, \|.,.\|)$.

Definition 3.1. We define the complex uncertain sequence $\{w_k\}$ as lacunary strongly convergent almost surely to w w.r.t the difference sequence in 2-normed space provided that for each $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n w_k(\gamma) - w(\gamma), z(\gamma)\| = 0,$$

for all $\gamma \in \Lambda$.

Definition 3.2. We define the complex uncertain sequence $\{w_k\}$ as lacunary strongly convergent in measure to w w.r.t the difference sequence in 2-normed space if

$$\lim_{r\to\infty} \mathcal{M}\left[\left\{\gamma \in \Gamma: \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n w_k(\gamma) - w(\gamma), z(\gamma)\| > \varepsilon\right\}\right] = 0,$$

for each $\varepsilon > 0$ and for all $z \neq 0$ in $(X, \|.,.\|)$.

Definition 3.3. We define the complex uncertain sequence $\{w_k\}$ as lacunary strongly convergent in mean to w w.r.t the difference sequence in 2-normed space if

$$\lim_{r\to\infty} E\left[\frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n w_k(\gamma) - w(\gamma), z(\gamma)\|\right] = 0,$$

for each $\varepsilon > 0$ and for all $z \neq 0$ in $(X, \|.,.\|)$.

Definition 3.4. Consider $\Phi_1, \Phi_2, \Phi_3, \ldots$ and so on, as the complex uncertainty distributions associated with the complex uncertain variables $\xi_1, \xi_2, \xi_3, \ldots$ and so forth. We define the complex uncertain $\{w_k\}$ as lacunary strongly convergent in distribution to w w.r.t the difference sequence in 2-normed space if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \|\triangle_m^n \Phi_k(c) - \Phi(c), z(\gamma)\| = 0,$$

for all $z \neq 0$ in $(X, \|.,.\|)$ and for all complex c at which $\Phi(c)$ is continuous.

Definition 3.5. The complex uncertain sequence $\{w_k\}$ is defined as be convergent uniformly almost surely to w if there exists an sequence of events $\{E_k\}$, $\mathcal{M}\{E_k\}$ approaches 0, such that $\{w_k\}$ converges uniformly to w in the complement of E_k, for any fixed $k \in \mathbb{N}$.

In the following discussion, we will explore the interconnections between various convergence concepts applicable to complex uncertain sequences, offering a renewed perspective on the topic.

Theorem 3.10. Suppose that the complex uncertain sequence $\{w_k\}$ exhibits lacunary strong convergence in terms of the difference sequence, leading to the limit w. In such a scenario, it can be inferred that the sequence $\{w_k\}$ also demonstrates lacunary strong convergence in measure towards the same limit w.

Proof. By utilizing Markov's inequality, we can establish that for any positive value ε, the following inequality holds true:

$$\lim_{r\to\infty}\mathcal{M}\left[\left\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma)-w(\gamma),\mathrm{z}(\gamma)\| > \varepsilon\right\}\right] \le \lim_{r\to\infty}\frac{E\left[\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma)-w(\gamma),\mathrm{z}(\gamma)\|\right]}{\varepsilon} \to 0$$

as $k \in I_r$. Therefore, we can conclude that the complex uncertain sequence $\{w_k\}$ exhibits lacunary strong convergence in measure w.r.t the difference sequence in 2-normed space, thereby establishing the proof of the theorem.

However, it should be noted that the converse of the aforementioned theorem is not universally valid. In other words, lacunary strong convergence in measure w.r.t the difference sequence does not necessarily imply lacunary strong convergence in mean w.r.t the difference sequence in 2-normed spaces. This fact can be demonstrated through the following example:

Let us consider the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\ldots\}$ with power set and

$$\mathcal{M}\{\Lambda\}=\begin{cases}\sup\limits_{\gamma_i\in\Lambda}\frac{1}{i}, & \text{if } \sup\limits_{\gamma_i\in\Lambda}\frac{1}{i}<0.5;\\ 1-\sup\limits_{\gamma_i\in\Lambda^c}\frac{1}{i}, & \text{if } \sup\limits_{\gamma_i\in\Lambda^c}\frac{1}{i}<0.5\\ 0.5, & \text{otherwise.}\end{cases}$$

and the complex uncertain variables be given as

$$w_i(\gamma_j)=\begin{cases} i, & \text{if } j=i\\ 0, & \text{if not}\end{cases}$$

for $i\in I_r$ and $L\equiv 0$.

For $\varepsilon>0$, we obtain

$$\lim_{r\to\infty}\mathcal{M}\left(\left\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma)-w(\gamma),\mathrm{z}(\gamma)\|>\varepsilon\right\}\right)=\lim_{r\to\infty}\mathcal{M}\left(\left\{\gamma\in\Gamma:\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma),\mathrm{z}(\gamma)\|>\varepsilon\right\}\right)=\lim_{r\to\infty}\mathcal{M}(\{\gamma_i\})=\lim_{r\to\infty}\frac{1}{i}\to 0\ (\text{ as } i\in I_r).$$

The sequence $\{w_k\}$ exhibits lacunary strong convergence in measure towards w. However, for each $i\in I_r$, the uncertainty distribution of the uncertain variable $\|w_k(\gamma)-w(\gamma),\mathrm{z}(\gamma)\|=\|w_k(\gamma),\mathrm{z}(\gamma)\|$ is

$$\Phi_i(x)=\begin{cases}0, & \text{if } x<0;\\ 1-\frac{1}{i}, & \text{if } 0\le x<i;\\ 1, & \text{if not.}\end{cases}$$

$$E\left[\frac{1}{h_r}\sum_{k\in I_r}\|\triangle_m^n w_k(\gamma)-w(\gamma),\mathrm{z}(\gamma)\|\right]=\int_0^{+\infty}\mathcal{M}\{w\ge x\}dx-\int_{-\infty}^{0}=1-\left(1-\frac{1}{i}\right)dx=1.$$

In other words, the $\{w_i(\gamma)\}$ does not converge in mean to $w(\gamma)$ w.r.t difference sequence.

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