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Power inequalities for the Berezin radius of operators in functional Hilbert spaces

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Abstract *–*A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Upsilon \subseteq \mathbb{C}$ that the evaluation functionals $\varphi_\varsigma(f) = f(\varsigma)$, $\varsigma \in \Upsilon$ are continuous on $\mathcal{H}$. The Berezin number of an operator X is defined by $ber(X) = sup_{\varsigma\in\Upsilon}\left|\tilde{X}(\varsigma)\right| = sup_{\xi\in\Upsilon}\left|\langle XK_\varsigma, K_\varsigma\rangle\right|$, where the operator X acts on the functional Hilbert space $\mathcal{H} = \mathcal{H}(\Upsilon)$ over some (non-empty) set Υ. We get a few Berezin number inequalities in this paper that build on some past findings in the literature. We also discover other associated inequalities.

Keywords – Berezin Transform, Bounded Operators, Functional Hilbert Space

I. Introduction

Berezin transformations have played a key role in several operator theory-related fields, including mathematical inequalities, mathematical analysis, numerical analysis, applied mathematics, and mathematical physics, to mention a few. To describe the Berezin number and norm, we start by discussing certain concepts and characteristics of operators on a Hilbert space.

Let $\mathcal{H}$ be a complex Hilbert space. A functional Hilbert space (briefly, FHS) $\mathcal{H} = \mathcal{H}(\mathcal{F})$ is a Hilbert space complex-valued functions on some set on some set $\Upsilon \subseteq \mathbb{C}$ that the evaluation functionals $\varphi_\varsigma(f) = f(\varsigma)$, $\varsigma \in \Upsilon$ are continuous on $\mathcal{H}$. Then by Riesz representation thorem, for each $\xi \in \Upsilon$ there is an unique element $k_\varsigma(\xi) \in \mathcal{H}$ such that $f(\varsigma) = \langle f, k_\varsigma\rangle$, for all $f \in \mathcal{H}$. The function k on $\Upsilon \times \Upsilon$ defined by $k_\varsigma(\xi) = k(\xi,\varsigma)$ is called the reproducing kernel of $\mathcal{H}$. Let $\mathfrak{I}_\varsigma = \frac{k_\varsigma}{\|k_\varsigma\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. The Berezin symbol (or Berezin transform) of the operator $X \in \mathbb{B}(\mathcal{H})$ is the bounded function $\tilde{X}$ on Υ defined by

$$\tilde{X}(\varsigma) = \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle, \varsigma \in \Upsilon,$$

(see, [1]), where $\mathbb{B}(\mathcal{H})$ is defined as the C^*-algebra of all bounded linear operators on $\mathcal{H}$. The Berezin set and Berezin number of an operator X are defined by

$$Ber(X) = \{\tilde{X}(\varsigma) : \varsigma \in \Upsilon\}$$

and

$$ber(X) = sup\{\left|\tilde{X}(\varsigma)\right| : \varsigma \in \Upsilon\} = sup_{\xi\in\Upsilon}\left|\langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right|$$

respectively (see, [15]). For more details on the Berezin symbol, we recommend the reader to [3, 7, 10, 11].

The Berezin range and Berezin number of an operator X in a FHS are, respectively, a subset of X's numerical range and numerical radius (i.e., $w(X) = sup\{|\langle Xu, u\rangle|,\ \|u\| = 1\}$). We refer to [8, 16] for numerical radius for fundamental characteristics. It is significant that

$$ber(X) \le w(X) \le \|X\|. \quad (1.1)$$

It is commonly known that

$$ber(X) \le \frac{1}{2}\left(\|X\| + \|X^2\|^{\frac{1}{2}}\right) \quad (1.2)$$

for every $X \in \mathbb{B}(\mathcal{H})$ (see [11, Theorem 4]). Huban et al. strengthened the second inequality in (1.2) using the Cartesian decomposition for operators in [13] by doing the following:

$$\frac{1}{4}\|X^*X+XX^*\|_{ber} \leq ber^2(X) \leq \frac{1}{2}\|X^*X+XX^*\|_{ber.} \tag{1.3}$$

for any operator $X \in \mathbb{B}(\mathcal{H})$ The same authors have also demonstrated that

$$\left(ber(X)\right)^{\varepsilon} \leq \frac{1}{2}\left\||X|^{2\gamma\varepsilon} + |X^*|^{2(1-\gamma)\varepsilon}\right\|_{ber} \tag{1.4}$$

and

$$\left(ber(X)\right)^{2\varepsilon} \leq \|\gamma|X|^{2\varepsilon} + (1-\gamma)|X^*|^{2\varepsilon}\|_{ber}$$

for $0 < \gamma < 1$ and $\varepsilon \geq 1$ (see, [14, Th. 3.1 and 3.2]). Recently, Gürdal [11, Theorem 1] generalized some inequalities for powers of the Berezin radius. See the most recent studies on Berezin radii inequalities [9, 12] and Berezin radius inequalities of block matrix [2, 4] for more results in this area.

II. MAIN RESULTS

The results that follow are important in and of themselves.

Theorem 1. *Let* $X, Y, Z \in \mathbb{B}(\mathcal{H})$ *with* $X, Y \geq 0$, *the operator matrix*

$$\begin{bmatrix} X & Z^* \\ Z & Y \end{bmatrix} \in \mathbb{B}(\mathcal{H}\oplus\mathcal{H})$$

be positive and $\rho, \sigma > 1$ *with* $\frac{1}{\rho}+\frac{1}{\sigma}=1$.

(i) If $\varepsilon > 0$ *and* $\rho\varepsilon, \sigma\varepsilon \geq 1$, *then*

$$ber^{2\varepsilon}(Z) \leq \left\|\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right\|_{ber} \tag{2.1}$$

(ii) If $\varepsilon \geq 1$ *and* $\rho\varepsilon, \sigma\varepsilon \geq 2$, *then*

$$ber^{2\varepsilon}(Z) \leq \frac{1}{2}\left[\|X\|_{Ber}^{\varepsilon}\|Y\|_{Ber}^{\varepsilon} + ber^{\varepsilon}(YX)\right] \tag{2.2}$$

and

$$ber^{2\varepsilon}(Z) \leq \frac{1}{2}\left(\left\|\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right\|_{ber} + ber^{\varepsilon}(YX)\right) \tag{2.3}$$

Proof. Let $\varsigma \in \Upsilon$ be an arbitary. From the inequalities given by [16, Lemma 1, p. 2], we have

$$\left|\langle Z\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^2 \leq \langle X\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\langle Y\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle.$$

By using the the Hölder-McCarthy inequality in [17, p. 20], we get

$$\begin{aligned}
\left|\langle Z\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^2 &\leq \langle X\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle^{\varepsilon}\langle Y\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle^{\varepsilon} \\
&\leq \frac{1}{\rho}\langle X\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle^{\rho\varepsilon} + \frac{1}{\sigma}\langle Y\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle^{\sigma\varepsilon} \\
&\leq \frac{1}{\rho}\langle X^{\rho\varepsilon}\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle + \frac{1}{\sigma}\langle Y^{\sigma\varepsilon}\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle \\
&= \langle\left(\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right)\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle
\end{aligned}$$

for $\varepsilon > 0$, and

$$sup_{\varsigma\in\Upsilon}\left|\langle Z\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^{2\varepsilon} \leq sup_{\varsigma\in\Upsilon}\langle\left(\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right)\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle$$

We deduce that

$$\begin{aligned}
ber^{2\varepsilon}(Z) = sup_{\varsigma\in\Upsilon}\left|\langle Z\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^{2\varepsilon} &\leq \\
&\leq sup_{\varsigma\in\Upsilon}\langle\left(\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right)\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle \\
&= \left\|\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right\|_{ber}
\end{aligned}$$

which proves (2.1). So, by using extension of the celebrated Cauchy-Schwarz inequality in [6, p. 20], we get

$$\begin{aligned}
\left|\langle Z\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^2 &\leq \langle X\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\langle Y\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle \\
&\leq \frac{\|X\mathfrak{J}_{\varsigma}\|\|Y\mathfrak{J}_{\varsigma}\| + \left|\langle X\mathfrak{J}_{\varsigma}, Y\mathfrak{J}_{\varsigma}\rangle\right|}{2}
\end{aligned}$$

and for $\varepsilon \geq 1$

$$\begin{aligned}
&\left|\langle Z\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^{2\varepsilon} \\
&\leq \left(\frac{\|X\mathfrak{J}_{\varsigma}\|\|Y\mathfrak{J}_{\varsigma}\| + \left|\langle X\mathfrak{J}_{\varsigma}, Y\mathfrak{J}_{\varsigma}\rangle\right|}{2}\right)^{\varepsilon} \\
&\leq \frac{\|X\mathfrak{J}_{\varsigma}\|^{\varepsilon}\|Y\mathfrak{J}_{\varsigma}\|^{\varepsilon} + \left|\langle X\mathfrak{J}_{\varsigma}, Y\mathfrak{J}_{\varsigma}\rangle\right|^{\varepsilon}}{2} \\
&= \frac{\|X\mathfrak{J}_{\varsigma}\|^{\varepsilon}\|Y\mathfrak{J}_{\varsigma}\|^{\varepsilon} + \left|\langle YX\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^{\varepsilon}}{2}
\end{aligned} \tag{2.4}$$

Taking the supremum over $\varsigma \in \Upsilon$ in the above inequality, we deduce

$$\begin{aligned}
&sup_{\varsigma\in\Upsilon}\left|\langle Z\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^{2\varepsilon} \\
&\leq sup_{\varsigma\in\Upsilon}\left(\frac{\|X\mathfrak{J}_{\varsigma}\|^{\varepsilon}\|Y\mathfrak{J}_{\varsigma}\|^{\varepsilon} + \left|\langle X\mathfrak{J}_{\varsigma}, Y\mathfrak{J}_{\varsigma}\rangle\right|^{\varepsilon}}{2}\right) \\
&\leq \frac{1}{2}\left(sup_{\varsigma\in\Upsilon}\left\{\|X\mathfrak{J}_{\varsigma}\|^{\varepsilon}\|Y\mathfrak{J}_{\varsigma}\|^{\varepsilon}\right\} + sup_{\varsigma\in\Upsilon}\left|\langle YX\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^{\varepsilon}\right) \\
&\leq \frac{1}{2}\left(sup_{\varsigma\in\Upsilon}\|X\mathfrak{J}_{\varsigma}\|^{\varepsilon}sup_{\varsigma\in\Upsilon}\|Y\mathfrak{J}_{\varsigma}\|^{\varepsilon} + sup_{\varsigma\in\Upsilon}\left|\langle YX\mathfrak{J}_{\varsigma}, \mathfrak{J}_{\varsigma}\rangle\right|^{\varepsilon}\right)
\end{aligned}$$

which is equivalent to

$$ber^{2\varepsilon}(Z) \le \frac{1}{2}\left(\|X\|_{Ber}^{\varepsilon}\|Y\|_{Ber}^{\varepsilon} + ber^{\varepsilon}(YX)\right)$$

This inequality gives the inequality (2.2). By using the inequality (2.4), we get

$$\begin{aligned}
\left|\langle Z\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle\right|^{2\varepsilon} &\le \frac{1}{2}\left(\left\|X\mathfrak{J}_\varsigma\right\|^{\varepsilon}\left\|Y\mathfrak{J}_\varsigma\right\|^{\varepsilon} + \left|\langle X\mathfrak{J}_\varsigma, Y\mathfrak{J}_\varsigma\rangle\right|^{\varepsilon}\right) \\
&\le \frac{1}{2}\left(\frac{1}{\rho}\left\|X\mathfrak{J}_\varsigma\right\|^{p\varepsilon} + \frac{1}{\sigma}\left\|Y\mathfrak{J}_\varsigma\right\|^{\sigma\varepsilon} + \left|\langle X\mathfrak{J}_\varsigma, Y\mathfrak{J}_\varsigma\rangle\right|^{\varepsilon}\right) \\
&= \frac{1}{2}\left(\frac{1}{\rho}\left\|X\mathfrak{J}_\varsigma\right\|^{2\frac{p\varepsilon}{2}} + \frac{1}{\sigma}\left\|Y\mathfrak{J}_\varsigma\right\|^{2\frac{\sigma\varepsilon}{2}} + \left|\langle X\mathfrak{J}_\varsigma, Y\mathfrak{J}_\varsigma\rangle\right|^{\varepsilon}\right) \\
&= \frac{1}{2}\left(\frac{1}{\rho}\langle X^2\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle^{\frac{\rho\varepsilon}{2}} + \frac{1}{\sigma}\langle Y^2\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle^{\frac{\sigma\varepsilon}{2}} + \left|\langle X\mathfrak{J}_\varsigma, Y\mathfrak{J}_\varsigma\rangle\right|^{\varepsilon}\right) \\
&\le \frac{1}{2}\left(\frac{1}{\rho}\langle X^{\rho\varepsilon}\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle + \frac{1}{\sigma}\langle Y^{\sigma\varepsilon}\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle + \left|\langle X\mathfrak{J}_\varsigma, Y\mathfrak{J}_\varsigma\rangle\right|^{\varepsilon}\right) \\
&= \frac{1}{2}\left(\langle\left(\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right)\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle + \left|\langle X\mathfrak{J}_\varsigma, Y\mathfrak{J}_\varsigma\rangle\right|^{\varepsilon}\right)
\end{aligned}$$

and, we reach that

$$sup_{\varsigma\in\Upsilon}\left|\langle Z\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle\right|^{2\varepsilon} \le \frac{1}{2}sup_{\varsigma\in\Upsilon}\left(\langle\left(\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right)\mathfrak{J}_\varsigma, \mathfrak{J}_\varsigma\rangle + \left|\langle X\mathfrak{J}_\varsigma, Y\mathfrak{J}_\varsigma\rangle\right|^{\varepsilon}\right)$$

which implies that

$$ber^{2\varepsilon}(Z) \le \frac{1}{2}\left(\left\|\frac{1}{\rho}X^{\rho\varepsilon} + \frac{1}{\sigma}Y^{\sigma\varepsilon}\right\|_{ber} + ber^{\varepsilon}(YX)\right)$$

as required.

With special choices ρ, σ and ε in Theorem 1, we get the following result.

Corollary 1. *(i)* $ber^{2\varepsilon}(Z) \le \frac{1}{2}\|X^{2\varepsilon} + Y^{2\varepsilon}\|_{ber}$ *for* $\varepsilon \ge \frac{1}{2}$ *and* $\rho = \sigma = 2$ *in (2.1),*
(ii) $ber(Z) \le \frac{1}{2}\|X + Y\|_{ber}$ *for* $\varepsilon = \frac{1}{2}$ *and* $\rho = \sigma = 2$ *in (2.1),*
(iii) $ber^2(Z) \le \frac{1}{2}\|X^2 + Y^2\|_{ber}$ *for* $\varepsilon = 1$ *and* $\rho = \sigma = 2$ *in (2.1) (see [5]),*
(iv) $ber^2(Z) \le \left\|\frac{1}{\rho}X^{\rho} + \frac{1}{\sigma}Y^{\sigma}\right\|_{ber}$ *for* $\varepsilon = 1$ *in (2.1) (see [5]).*
(v) $ber^2(Z) \le \frac{1}{2}[\|X\|_{Ber}\|Y\|_{Ber} + ber(YX)]$ *for* $\varepsilon = 1$ *in (2.2) (see [5])*
(vi) $ber^4(Z) \le \frac{1}{2}[\|X\|_{Ber}^2\|Y\|_{Ber}^2 + ber^2(YX)]$ *for* $\varepsilon = 2$ *in (2.2),*
(vii) $ber^{2\varepsilon}(Z) \le \frac{1}{2}\left(\frac{1}{2}\|X^{2\varepsilon} + Y^{2\varepsilon}\|_{ber} + ber^{\varepsilon}(YX)\right)$ *for* $\varepsilon \ge 1$ *and* $\rho = \sigma = 2$ *in (2.3).*
(viii) $ber^4(Z) \le \frac{1}{2}\left(\left\|\frac{1}{\rho}X^{2\rho} + \frac{1}{\sigma}Y^{2\sigma}\right\|_{ber} + ber^2(YX)\right)$ *for* $\varepsilon = 2$ *and* $\rho, \sigma > 1$ *in(2.3).*
(ix) $ber^4(Z) \le \frac{1}{2}\left(\frac{1}{2}\|X^4 + Y^4\|_{ber} + ber^2(YX)\right)$ *for* $\rho = \sigma = \varepsilon = 2$ *in (2.3).*

Assume that $\begin{bmatrix} AA^* & AB \\ B^*A^* & B^*B \end{bmatrix} \in \mathbb{B}(\mathcal{H}\oplus\mathcal{H})$. Putting $X = |A^*|^2$, $Y = |B|^2$ and $Z = B^*A^*$ in Theorem 1, we obtain the following theorem.

Theorem 2. *Let* $A, B \in \mathbb{B}(\mathcal{H})$ *and* $\rho, \sigma > 1$ *with* $\frac{1}{\rho} + \frac{1}{\sigma} = 1$. *Then*
(i) If $\varepsilon > 0$ *and* $\rho\varepsilon, \sigma\varepsilon \ge 1$, *then*

$$ber^{2\varepsilon}(AB) \le \left\|\frac{1}{\rho}|A^*|^{2\rho\varepsilon} + \frac{1}{\sigma}|B|^{2\sigma\varepsilon}\right\|_{ber} \qquad (2.5)$$

(ii) If $\varepsilon \ge 1$, *then*

$$ber^{2\varepsilon}(AB) \le \frac{1}{2}[\|A\|_{Ber}^{2\varepsilon}\|B\|_{Ber}^{2\varepsilon} + ber^{\varepsilon}(|B|^2|A^*|^2)] \qquad (2.6)$$

(iii) If $\varepsilon \ge 1$ *and* $\rho\varepsilon, \sigma\varepsilon \ge 2$, *then*

$$ber^{2\varepsilon}(AB) \le \frac{1}{2}\left(\left\|\frac{1}{\rho}|A^*|^{2\rho\varepsilon} + \frac{1}{\sigma}|B|^{2\sigma\varepsilon}\right\|_{ber} + ber^{\varepsilon}(|B|^2|A^*|^2)\right) \qquad (2.7)$$

With special choices ρ, σ and ε in Theorem 2, we get the following result.

Corollary 2. *(i)* $ber^{2\varepsilon}(AB) \le \frac{1}{2}\||A^*|^{4\varepsilon} + |B|^{4\varepsilon}\|_{ber}$ *for* $\varepsilon \ge \frac{1}{2}$ *and* $\rho = \sigma = 2$ *in (2.5),*
(ii) $ber(AB) \le \frac{1}{2}\||A^*|^2 + |B|^2\|_{ber}$ *for* $\varepsilon = \frac{1}{2}$ *in (2.5),*
(iii) $ber^2(AB) \le \frac{1}{2}\||A^*|^4 + |B|^4\|_{ber}$ *for* $\varepsilon = 1$ *in*

(2.5) (see [5]),
(iv) $ber^2(AB) \leq \left\| \frac{1}{\rho}|A^*|^{2\rho} + \frac{1}{\sigma}|B|^{2\sigma} \right\|_{ber}$ *for* $\varepsilon = 1$ *and* $\rho, \sigma > 1$ *in (2.5),*
(v) $ber^4(AB) \leq \frac{1}{2}[\|A\|_{Ber}^4 \|B\|_{Ber}^4 + ber^2(|B|^2|A^*|^2)]$ *for* $\varepsilon = 2$ *in (2.6),*
(vi) $ber^{2\varepsilon}(AB) \leq \frac{1}{2}\left(\frac{1}{2}\||A^*|^{4\varepsilon} + |B|^{4\varepsilon}\|_{ber} + ber^{\varepsilon}(|B|^2|A^*|^2)\right)$ *for* $\varepsilon \geq 1$ *and* $\rho = \sigma = 2$ *in (2.7),*
(vii) $ber^4(AB) \leq \frac{1}{2}\left(\left\| \frac{1}{\rho}|A^*|^{4\rho} + \frac{1}{\sigma}|B|^{4\sigma} \right\|_{ber} + ber^2(|B|^2|A^*|^2)\right)$ *for* $\varepsilon = 2$ *and* $\rho, \sigma > 1$ *in (2.7),*
(viii) $ber^4(AB) \leq \frac{1}{2}\left(\frac{1}{2} \left\| |A^*|^8 + \frac{1}{\sigma}|B|^8 \right\|_{ber} + ber^2(|B|^2|A^*|^2)\right)$ *for* $\rho = \sigma = \varepsilon = 2$ *in (2.7).*

Now, we will prove the following theorem.

Theorem 3. *Let* $X, Y, Z \in \mathbb{B}(\mathcal{H})$ *with* $X, Y \geq 0$ *and the operator matrix*

$$\begin{bmatrix} X & Z^* \\ Z & Y \end{bmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$$

be positive. Then we have

$$ber^2(Z) \leq \|(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \|X\|_{ber}^{\gamma} \|Y\|_{ber}^{1-\gamma} \tag{2.8}$$

and

$$ber^2(Z) \leq \|(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \|\gamma X^{\varepsilon} + (1-\gamma)Y^{\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \tag{2.9}$$

for $\gamma \in [0,1]$ *and* $\varepsilon \geq 1$.

Proof. Let $\varsigma \in \Upsilon$ be an arbitary. From the inequalities given by [16, Lemma 1, p. 2] and the McCarthy inequality for $\varepsilon \geq 1$, we get

$$\begin{aligned} \left|\langle Z\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right|^2 &\leq \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \\ &= \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{1-\gamma} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\gamma} \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\gamma} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{1-\gamma} \\ &\leq \left[(1-\gamma)\langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle + \gamma\langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right] \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\gamma} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{1-\gamma} \end{aligned}$$

and

$$\begin{aligned} (1-\gamma)\langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon} + \gamma\langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon} &\leq (1-\gamma)\langle X^{\varepsilon}\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle + \gamma\langle Y^{\varepsilon}\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \\ &= \langle [(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}]\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \end{aligned}$$

For $\varepsilon \geq 1$, by using the convexity, we reach that

$$\begin{aligned} \left|\langle Z\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right|^{2\varepsilon} &\leq \left[(1-\gamma)\langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle + \gamma\langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right]^{\varepsilon} \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon X} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon(1-\gamma)} \\ &\leq \left[(1-\gamma)\langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon} + \gamma\langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon}\right] \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon X} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon(1-\gamma)} \\ &\leq \langle [(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}]\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon X} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon(1-\gamma)} \end{aligned}$$

and, so

$$\begin{aligned} ber^{2\varepsilon}(Z) &= sup_{\varsigma \in \Upsilon}\left|\langle Z\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right|^{2\varepsilon} \\ &\leq sup_{\varsigma \in \Upsilon}\left\{\langle [(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}]\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon X} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\varepsilon(1-\gamma)}\right\} \\ &= \|(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}\|_{ber} \|X\|_{ber}^{\varepsilon X} \|Y\|_{ber}^{\varepsilon(1-\gamma)} \end{aligned}$$

as required the inequality (2.8). Using similar arguments above, we deduce that

$$\begin{aligned} \left|\langle Z\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right|^2 &\leq \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \\ &= \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{1-\gamma} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\gamma} \langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{\gamma} \langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle^{1-\gamma} \\ &\leq \left[(1-\gamma)\langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle + \gamma\langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right]\left[\gamma\langle X\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle + (1-\gamma)\langle Y\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right] \end{aligned}$$

and

$$\left|\langle Z\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle\right|^{2\varepsilon} \leq \langle [(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}]\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle \langle [\gamma X^{\varepsilon} + (1-\gamma)Y^{\varepsilon}]\mathfrak{I}_\varsigma, \mathfrak{I}_\varsigma\rangle$$

Hence, by taking the supremum over $\varsigma \in \Upsilon$ in the above inequality, we reach that

$$ber^2(Z) \leq \|(1-\gamma)X^{\varepsilon} + \gamma Y^{\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \|\gamma X^{\varepsilon} + (1-\gamma)Y^{\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}$$

which proves the inequality (2.9).

From Theorem 3, then we have the following result.

Corollary 3. *We have*

$$ber^2(Z) \leq \frac{1}{2^{\frac{1}{\varepsilon}}} \|X^{\varepsilon} + \gamma Y^{\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \|X\|^{\frac{1}{2}} \|Y\|_{ber}^{\frac{1}{2}} \tag{2.10}$$

and

$$ber(Z) \leq \frac{1}{2^{\frac{1}{\varepsilon}}} \|X^{\varepsilon} + \gamma Y^{\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \tag{2.11}$$

If we put $X = |A^*|^2$, $Y = |B|^2$ and $Z = B^*A^*$, in Theorem 3, then we have the following result.

Theorem 4. *Let* $A, B \in \mathbb{B}(\mathcal{H})$, *then for* $\gamma \in [0,1]$ *and* $\varepsilon \geq 1$,

$$ber^2(AB) \leq \|(1-\gamma)|A^*|^{2\varepsilon} + \gamma|B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \|A\|_{ber}^{2\gamma} \|B\|_{ber}^{2(1-\gamma)} \quad (2.12)$$

and

$$ber^2(AB) \leq \|(1-\gamma)|A^*|^{2\varepsilon} + \gamma|B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \|\gamma|A^*|^{2\varepsilon} + (1-\gamma)|B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \quad (2.13)$$

In particular,

$$ber^2(AB) \leq \frac{1}{2^{\frac{1}{\varepsilon}}} \||A^*|^{2\varepsilon} + |B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}} \|A\|_{Ber} \|B\|_{Ber} \quad (2.14)$$

and

$$ber(AB) \leq \frac{1}{2^{\frac{1}{\varepsilon}}} \||A^*|^{2\varepsilon} + |B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}$$

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