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Bohr Radius For A Certain Subclass Of Harmonic Functions Defined By A New Family

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Abstract – In this article, we obtain Bohr radius for the subclass $\mathcal{R}_{\mathcal{H}}^{n}(\alpha,\gamma,\beta)=\{f=h+\bar{g}:\mathrm{Re}[h'(z)+\alpha zh''(z)+\gamma z^2h'''(z)-\beta]>|g'(z)+\alpha zg''(z)+\gamma z^2g'''(z)|\}$, where $h(z)=z+\sum_{k=n+1}^{\infty}a_kz^k$, $g(z)=\sum_{k=n+1}^{\infty}b_kz^k$ are analytic in the open unit disk, and $\alpha\geq\gamma\geq 0$, $0\leq\beta<1$ and $n\geq 1$.

Keywords – Bohr Radius, Harmonic Functions, Univalent Functions

I. Introduction

Let $\mathcal{H}$ be the class of complex-valued harmonic functions f in $U=\{z\in\mathbb{C}:\ |z|<1\}$, normalized so that $f(0)=0$, $f_z(0)=1$. Also, let $\mathcal{H}_0=\{f\in\mathcal{H}:\ f_{\bar{z}}(0)=0\}$. Such an $f\in\mathcal{H}_0$ has the decomposition $f=h+\bar{g}$, where h and g are analytic in U and has the following representation:

$$h(z)=z+\sum_{k=2}^{\infty}a_kz^k,\quad g(z)=\sum_{k=2}^{\infty}b_kz^k. \qquad (1)$$

A harmonic function f is locally univalent and sense-preserving in U if and only if $J_f(z)=|f_z(z)|^2-|f_{\bar{z}}(z)|^2$ is positive in U. Set

$$\mathcal{H}_0^n=\{f=h+\bar{g}\in\mathcal{H}:h'(0)-1=g'(0)=h''(0)=\cdots=h^{(n)}(0)=g^{(n)}(0)=0\},$$

where $n\geq 2$. When n = 1, we have $\mathcal{H}_0^1\equiv\mathcal{H}_0$. Thus, each $f=h+\bar{g}\in\mathcal{H}_0^n$ has the form

$$h(z)=z+\sum_{k=n+1}^{\infty}a_kz^k,$$

and

$$g(z)=\sum_{k=n+1}^{\infty}b_kz^k. \qquad (2)$$

See [2,4]. In [7], Ponnusamy et. al. introduced a class $P_{\mathcal{H}}^0:=\{f\in\mathcal{H}_0:Re[h'(z)]>|g'(z)|\}$ for $z\in U$ and they proved that functions in $P_{\mathcal{H}}^0$ are univalent in U.

In [6], Nagpal and Ravichandran studied a class $W_{\mathcal{H}}^0$ of functions $f\in\mathcal{H}_0$ satisfying the condition $Re[h'(z)+zh''(z)]>|g'(z)+zg''(z)|$ for $z\in U$.

Ghosh and Vasudevarao [5] investigated the class $W_{\mathcal{H}}^0(\alpha)$ of functions $f\in\mathcal{H}_0$ satisfying the condition $Re[h'(z)+\alpha zh''(z)]>|g'(z)+\alpha zg''(z)|$ for $0\leq\alpha$, and $z\in U$.

Yaşar and Yalçın defined $\mathcal{R}_{\mathcal{H}}^0(\alpha,\gamma)$ class functions $f=h+\bar{g}\in\mathcal{H}_0$ and satisfy

$$\mathrm{Re}[h'(z)+\alpha zh''(z)+\gamma z^2h'''(z)]>|g'(z)+\alpha zg''(z)+\gamma z^2g'''(z)| \qquad (3)$$

where $\alpha\geq\gamma\geq 0$ (See [8]).

Denote by $\mathcal{R}_{\mathcal{H}}^n(\alpha,\gamma,\beta)$, the class of functions $f=h+\bar{g}\in\mathcal{H}^0$ and satisfy

$$\mathrm{Re}[h'(z)+\alpha zh''(z)+\gamma z^2h'''(z)-\beta]>|g'(z)+\alpha zg''(z)+\gamma z^2g'''(z)| \qquad (3)$$

where $\alpha \geq \gamma \geq 0$, $0 \leq \beta < 1$ and $n \geq 1$ (See [3]).

The class $\mathcal{R}^1_{\mathcal{H}}(\alpha,\gamma,\beta) = \mathcal{R}_{\mathcal{H}}(\alpha,\gamma,\beta)$ generalizes several previously studied classes of harmonic mappings. For examples, $\mathcal{R}_{\mathcal{H}}(0,0,0) = \mathcal{P}^0_{\mathcal{H}}$ [9], $\mathcal{R}_{\mathcal{H}}(1,0,0) = \mathcal{W}^0_{\mathcal{H}}$ [10], $\mathcal{R}_{\mathcal{H}}(\alpha,0,0) = \mathcal{W}^0_{\mathcal{H}}(\alpha)$ [5], $\mathcal{R}_{\mathcal{H}}(0,0,\beta) = \mathcal{P}^0_{\mathcal{H}}(\beta)$ [7] and $\mathcal{R}_{\mathcal{H}}(\alpha,\gamma,0) = \mathcal{R}^0_{\mathcal{H}}(\alpha,\gamma)$ [8]. We denote $\mathcal{R}^n_{\mathcal{H}}(\alpha,\gamma,0) = \mathcal{R}^n_{\mathcal{H}}(\alpha,\gamma)$ and $\mathcal{R}^1_{\mathcal{H}}(\alpha,\gamma,\beta) = \mathcal{R}_{\mathcal{H}}(\alpha,\gamma,\beta)$.

Definition 1. Let $f = h + \bar{g} \in \mathcal{H}^0$ be a harmonic function h and g are given by (1). Then the Bohr Phenomenon is to find the constant $0 < \rho_* \leq 1$ such that the inequality

$$\rho + \sum_{k=2}^{\infty}(|a_k| + |b_k|)\,\rho^k \leq d\left(f(0), \partial\left(f(U)\right)\right)$$

holds for all $|z| = \rho \leq \rho_*$, where $d\left(f(0), \partial\left(f(U)\right)\right)$ denotes the Euclidean distance between $f(0)$ and the boundary of $f(U)$. The largest such ρ_* is called the Bohr radius.

The idea of Bohr radius, originated from the work of Bohr (see [1]) on the inequality $\sum_{k=2}^{\infty}|a_k|\rho^k \leq 1$ ($\rho \leq 1/3$) for an analytic function with the power series $\sum_{k=0}^{\infty} a_k z^k$, which is known as Bohr's Theorem. Finding the Bohr radius for such inequalities with diverse possibilities has become a popular topic.

II. Bohr Radius For The Class $\mathcal{R}^n_{\mathcal{H}}(\alpha,\gamma,\beta)$

Lemma 1. [3] Suppose $f \in \mathcal{R}^n_{\mathcal{H}}(\alpha,\gamma,\beta)$. Then for $n \geq 1$ and $k \geq n+1$,

$$|a_k| + |b_k| \leq \frac{2(1-\beta)}{k[1 + (k-1)\alpha + (k^2 - 3k + 2)\gamma]}.$$

Lemma 2. [3] Suppose $f \in R^n_{\mathcal{H}}(\alpha,\gamma,\beta)$. Then

$$|z| + \sum_{k=n+1}^{\infty} \frac{2(1-\beta)(-1)^{k-1}|z|^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)} \leq |f(z)|.$$

Theorem 1. $f \in \mathcal{R}^n_{\mathcal{H}}(\alpha,\gamma,\beta)$. Then

$$|z| + \sum_{k=n+1}^{\infty} (|a_k| + |b_k|)|z|^k \leq d\left(f(0), \partial\left(f(U)\right)\right)$$

for $|z| < \rho_*$, where ρ_* is the unique positive root in (0,1) of

$$\rho + \sum_{k=n+1}^{\infty} \frac{2(1-\beta)\rho^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)} = 1 - \sum_{k=n+1}^{\infty} \frac{2(1-\beta)(-1)^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)}.$$

The radius ρ_* is the Bohr radius fort he class $\mathcal{R}^n_{\mathcal{H}}(\alpha,\gamma,\beta)$.

Proof. From Lemma 2, it follows that the distance between origin and the boundary of $f(U)$ satisfies

$$d\left(f(0), \partial\left(f(U)\right)\right) \geq 1 - \sum_{k=n+1}^{\infty} \frac{2(1-\beta)(-1)^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)}. \quad (3)$$

Let consider the continuous function

$$\Phi(\rho) = \rho + \sum_{k=n+1}^{\infty} \frac{2(1-\beta)\rho^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)} - 1 + \sum_{k=n+1}^{\infty} \frac{2(1-\beta)(-1)^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)}.$$

Now

$$\Phi'(\rho) = 1 + 2(1-\beta)\sum_{k=n+1}^{\infty} \frac{\rho^{k-1}}{1 + (k-1)\alpha + (k-1)(k-2)\gamma} > 0$$

for all $\rho \in (0,1)$, which implies that Φ is a strictly increasing continuous function. Note that $\Phi(0) < 0$ and

$$\Phi(1) = 2(1-\beta)\sum_{k=n+1}^{\infty} \frac{1}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)} + 2\sum_{k=n+1}^{\infty} \frac{(-1)^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)} > 0.$$

Thus by Intermediate Value Theorem for continuous function, we let ρ_* be the unique root of $\Phi(\rho) = 0$ in (0,1). Now using Lemma 1 and the inequality (3), we have

$$|z| + \sum_{k=n+1}^{\infty} (|a_k| + |b_k|)|z|^k \leq \rho + \sum_{k=n+1}^{\infty} \frac{2(1-\beta)\rho^k}{k(1 + (k-1)\alpha + (k-1)(k-2)\gamma)}$$

$$\leq \rho_* + \sum_{k=n+1}^{\infty} \frac{2(1-\beta){\rho_*}^k}{k(1+(k-1)\alpha+(k-1)(k-2)\gamma)}$$

$$= 1 - \sum_{k=n+1}^{\infty} \frac{2(1-\beta)(-1)^k}{k(1+(k-1)\alpha+(k-1)(k-2)\gamma)}$$

$$\leq d\Big(f(0), \partial\big(f(U)\big)\Big),$$

which hold for $\rho \leq \rho_*$. Now consider the analytic function

$$f(z) = z + 2(1-\beta) \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k(1+(k-1)\alpha+(k-1)(k-2)\gamma)} z^k.$$

Then clearly $f \in \mathcal{R}_{\mathcal{H}}^n(\alpha,\gamma,\beta)$ and at $|z| = \rho_*$, we get

$$|z| + \sum_{k=n+1}^{\infty} (|a_k| + |b_k|)|z|^k = d\Big(f(0), \partial\big(f(U)\big)\Big).$$

Hence the radius ρ_* is the Bohr radius for the class $\mathcal{R}_{\mathcal{H}}^n(\alpha,\gamma,\beta)$.

Now using Theorem 1, we can obtain Bohr radius for the classes $\mathcal{R}_{\mathcal{H}}^0(0,0,0) \equiv P_{\mathcal{H}}^0$, $\mathcal{R}_{\mathcal{H}}^0(\alpha,0,0) \equiv W_{\mathcal{H}}^0(\alpha)$, $\mathcal{R}_{\mathcal{H}}^0(0,0,\beta) \equiv \mathcal{P}_{\mathcal{H}}^0(\beta)$ and $\mathcal{R}_{\mathcal{H}}(\alpha,\gamma,0) = \mathcal{R}_{\mathcal{H}}^0(\alpha,\gamma)$ Here we mention the following:

Corollary 1. $f \in P_{\mathcal{H}}^0$. Then

$$|z| + \sum_{k=2}^{\infty} (|a_k| + |b_k|)|z|^k \leq d\Big(f(0), \partial\big(f(U)\big)\Big)$$

for $|z| \leq \rho_*$, where ρ_* is the unique positive root in (0,1) of

$$\rho + 2\sum_{k=2}^{\infty} \frac{\rho^k}{k} = 1 - 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k}.$$

Corollary 2. $f \in W_{\mathcal{H}}^0(\alpha)$. Then

$$|z| + \sum_{k=2}^{\infty} (|a_k| + |b_k|)|z|^k \leq d\Big(f(0), \partial\big(f(U)\big)\Big)$$

for $|z| \leq \rho_*$, where ρ_* is the unique positive root in (0,1) of

$$\rho + 2\sum_{k=2}^{\infty} \frac{\rho^k}{\alpha k^2 + (1-\alpha)k} = 1 - 2\sum_{k=2}^{\infty} \frac{(-1)^k}{\alpha k^2 + (1-\alpha)k}.$$

Corollary 3. $f \in \mathcal{P}_{\mathcal{H}}^0(\beta)$. Then

$$|z| + \sum_{k=2}^{\infty} (|a_k| + |b_k|)|z|^k \leq d\Big(f(0), \partial\big(f(U)\big)\Big)$$

for $|z| \leq \rho_*$, where ρ_* is the unique positive root in (0,1) of

$$\rho + 2(1-\beta)\sum_{k=2}^{\infty} \frac{\rho^k}{k} = 1 - 2(1-\beta)\sum_{k=2}^{\infty} \frac{(-1)^k}{k}.$$

Corollary 4. $f \in R_{\mathcal{H}}^0(\alpha,\gamma)$. Then

$$|z| + \sum_{k=2}^{\infty} (|a_k| + |b_k|)|z|^k \leq d\Big(f(0), \partial\big(f(U)\big)\Big)$$

for $|z| < \rho_*$, where ρ_* is the unique positive root in (0,1) of

$$\rho + \sum_{k=2}^{\infty} \frac{2\rho^k}{\gamma k^3 + (\alpha-3\gamma)k^2 + (1-\alpha+2\gamma)k} = 1 - \sum_{k=2}^{\infty} \frac{2(-1)^k}{\gamma k^3 + (\alpha-3\gamma)k^2 + (1-\lambda+2\gamma)k}.$$

The radius ρ_* is the Bohr radius fort he class $R_{\mathcal{H}}^0(\alpha,\gamma)$.

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