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On some classes of triple sequence spaces defined by Orlicz function-I

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Abstract – In this conference paper, we introduce triple sequence spaces via Orlicz function and examine some properties of the resulting these spaces like as linear space, seminormed space and some inclusion relations.

Keywords – Pringsheim Limit, Triple Sequence Space,Convex Function, Continuous Function, Orlicz Function

I. INTRODUCTION

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

Recall in [11] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Ruckle [16].An Orlicz function M is said to satisfy $\Delta_2 - condition$ for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{(x_i): \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{r}\right) < \infty, \text{for some } r > 0\right\},$$

which is a Banach space normed by

$$\|(x_i)\| = \inf\left\{r > 0: \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{r}\right) \leq 1\right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta [5], Esi [1 − 2], Esi and Et [3], Parashar and Choudhary [12] and many others.

By the convergence of a triple sequence we mean the convergence on the Pringsheim sense that is, a triple sequence $x = (x_{ijl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{ijl} - L| < \varepsilon$ whenever $i, j, l > n$, [4]. We shall write more briefly as "$P - convergent$". The initial works on triple sequences is found in Bromwich [14]. Later on it was studied by Hardy [8], Moricz [6], Moricz and Rhoades [7] and many others. Hardy [8] introduced the notion of regular convergence for double sequences.

The concept of paranormed sequences was studied by Nakano [9] and Simons [13] at the initial stage. Later on it was studied by many others.

The triple sequence $x = (x_{ijl})$ is bounded if there exists a positive number M such that $|x_{ijl}| < M$ for all $i, j, l \in \mathbb{N}$. Let l_∞^3 the space of all bounded triple sequences such that

$$\|x_{ijl}\|_{(\infty,3)} = \sup_{i,j,l}|x_{ijl}| < \infty.$$

Throughout the paper, $w^3(X)$ denotes the spaces of all triple sequences in X, where (X,q) denotes a seminormed space, seminormed by q. The zero double sequence is denoted by θ in X.

II. *DEFINITIONS AND BACKGROUND*

Let P_s denotes the class of all subsets of $\mathbb{N}$, those do not contain more than s elements and $\{\phi_n\}$ represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all $n \in \mathbb{N}$.

The sequence space $m(\phi)$ introduced by Sargent [15] is defined as follows:

$$m(\phi) = \left\{x = (x_k): \|x_k\|_{m(\phi)} = \sup_{s\ge 1,\sigma\in P_s} \frac{1}{\phi_s}\sum_{i\in\sigma} |x_i| < \infty\right\}$$

and studied some of its properties and obtained its relationship with the space l_p.

A triple sequence space E is said to be solid or normal if $(\alpha_{ijl}x_{ijl}) \in E$, whenever $(x_{ijl}) \in E$ for all triple sequences (α_{ijl}) of scalars such that $|\alpha_{ijl}| \le 1$ for all $i,j,l \in \mathbb{N}$.

A triple sequence space E is said to be symmetric if $(x_{ijl}) \in E$ implies $(x_{\pi(i)\pi(j)\pi(l)}) \in E$, where π is a permutation of the elements of $\mathbb{N}$.

Let $K = \{(i_n, j_k, l_m): n,k,m \in \mathbb{N}; i_1 < i_2 < i_3 < \ldots, j_1 < j_2 < j_3 < \ldots \text{and} l_1 < l_2 < l_3 < \ldots\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and E be a triple sequence space. A $K-step\ space$ of E is a sequence space

$$\lambda_K^E = \{(x_{i_n j_k l_m}): (x_{ijl}) \in E\}.$$

A canonical pre-image of a sequence $(x_{ijl}) \in E$ is a sequence $(y_{ijl}) \in E$ defined as follows:

$$y_{ijl} = \begin{cases} x_{ijl}, & \text{if}(i,j,l) \in K \\ 0, & \text{otherwise} \end{cases}.$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A triple sequence space E is said to be monotone if E contains the canonical pre-images of all its step spaces.

Lemma. A double sequence space E is solid implies E is monotone.

Let P_{stq} denotes the class of all subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, those do not contain more that $s\times t\times q$ elements. Throughout the paper $\{\phi_{n,m,p}\}$ represent a non-decreasing triple sequence of real numbers such that $n\phi_{n+1,m,p} \le (n+1)\phi_{n,m,p}$ and $m\phi_{n,m+1,p} \le (m+1)\phi_{n,m,p}$ and $p\phi_{n,m,p+1} \le (p+1)\phi_{n,m,p}$. In this paper we introduce the following triple sequence spaces: Let M be an Orlicz function and a $p = (p_{ijl})$ be a bounded triple sequence of positive real numbers such that $0 < H_o = \inf_{i,j,l} p_{ijl} \le p_{ijl} \le \sup_{i,j,l} p_{ijl} = H < \infty$, then

$$l_\infty^3(M,q,p) = \left\{x = (x_{ijl}) \in w^3(X): \sup_{i,j,l}\left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty, \text{for some } r > 0\right\},$$

$$l_p^3(M,q) = \left\{x = (x_{ijl}) \in w^3(X): \sum_{i,j.l=1,1,1}^{\infty}\left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty, \text{for some } r > 0\right\},$$

$$m^3(M,\phi,q,p) = \left\{x = (x_{ijl}) \in w^3(X): \sup_{s,t,q\ge 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma}\left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty, \text{for some } > 0\right\}.$$

The following inequality will be used throughout the paper

$$|a_{ijl}+b_{ijl}|^{p_{ijl}} \le \max(1,2^{H-1})\left(|a_{ijl}|^{p_{ijl}}+|b_{ijl}|^{p_{ijl}}\right)$$

where a_{ijl} and b_{ijl} are complex numbers and $H = \sup_{i,j,l} p_{ijl} < \infty$.

It is easy to see that the triple sequence space $l_p^3(M,q)$ is a seminormed space, seminormed by

$$g\left(\left(x_{ijl}\right)\right) = \inf\left\{r^{\frac{p_{ijl}}{J}} > 0 : \sum_{i,j,l=1,1,1}^{\infty} M\left(q\left(\frac{x_{ijl}}{r}\right)\right) \le 1\right\}, \text{where } J = max(1,2^{H-1}).$$

III. *RESULTS*

In this section we continue to prove some results involving the triple sequence spaces $m^3(M,\phi,q,p), l_p^3(M,q)$ and $l_\infty^3(M,q,p)$ in this first part of this conference paper.

Theorem 3.6. Let M, M_1, M_2 be Orlicz functions satisfying $\Delta_2 - condition$. Then

(i) $m^3(M_1,\phi,q,p) \subset m^3(MoM_1,\phi,q,p)$,

(ii) $m^3(M_1,\phi,q,p) \cap m^3(M_2,\phi,q,p) \subset m^3(M_1+M_2,\phi,q,p)$.

Proof. **(i)** Let $(x_{ijl}) \in m^3(M_1,\phi,q,p)$. Then there exists $r > 0$ such that

$$\sup_{s,t,q\ge 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M_1\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty.$$

Let $0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \le t < \delta$. Let $y_{ij} = M_1\left(q\left(\frac{x_{ijl}}{r}\right)\right)$ and for any $\sigma \in P_{stq}$, let

$$\sum_{(i,j,l)\in\sigma} \left[M(y_{ijl})\right]^{p_{ijl}} = \sum_1 \left[M(y_{ijl})\right]^{p_{ijl}} + \sum_2 \left[M(y_{ijl})\right]^{p_{ijl}}$$

where the first summation is over $y_{ijl} \le \delta$ and the second is over $y_{ijl} > \delta$. By the remark we have

$$\sum_1 M(y_{ijl}) \le \max(1,[M(1)]^H) \sum_1 \left(y_{ijl}\right)^{p_{ilj}} \le \max(1,[M(2)]^H) \sum_1 \left(y_{ijl}\right)^{p_{ijl}} \qquad (3.1)$$

For $y_{ij} > \delta$

$$y_{ijl} < y_{ijl}\delta^{-1} \le 1 + y_{ijl}\delta^{-1},$$

since M is non-decreasing and convex, so

$$M(y_{ijl}) < M(1+y_{ijl}\delta^{-1}) < \tfrac{1}{2}M(2) + \tfrac{1}{2}M(2y_{ijl}\delta^{-1}).$$

Since M satisfies $\Delta_2 - condition$, so

$$M(y_{ijl}) < \tfrac{K}{2} y_{ijl}\delta^{-1}M(2) + \tfrac{K}{2} y_{ijl}\delta^{-1}M(2) = K y_{ijl}\delta^{-1}M(2).$$

Hence,

$$\sum_2 \left[M(y_{ijl})\right]^{p_{ijl}} \le \max(1,[K\delta^{-1}M(2)]^H) \sum_2 \left(y_{ijl}\right)^{p_{ijl}}. \qquad (3.2)$$

By (3.1) and (3.2) we have $(x_{ijl}) \in m^3(MoM_1,\phi,q,p)$. Thus $m^3(M_1,\phi,q,p) \subset m^3(MoM_1,\phi,q,p)$.

(ii) Let $(x_{ijl}) \in m^3(M_1,\phi,q,p) \cap m^3(M_2,\phi,q,p)$. Then there exists $r > 0$ such that

$$\sup_{s,t,q\ge 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M_1\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty$$

and

$$\sup_{s,t,q\ge 1,\sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M_2\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ilj}} < \infty.$$

The rest of the proof follows from the equality

$$\sum_{(i,j,l)\in\sigma} \left[(M_1+M_2)\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}}$$

$\leq \max(1, 2^{H-1}) \sum_{(i,j,l)\in\sigma} \left[M_1\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} + \max(1, 2^{H-1}) \sum_{(i,j,l)\in\sigma} \left[M_2\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}}.$

This completes the proof.

Taking $M_1(x) = x$ in above theorem, we have the following result.

Corollary 3.7. Let M be an Orlicz function satisfying $\Delta_2 - condition$, then $m^3(\phi, q, p) \subset m^3(M, \phi, q, p)$.

From Theorem 3.4. and Corollary 3.7., we have:

Corollary 3.8. Let M be an Orlicz function satisfying $\Delta_2 - condition$, then $m^3(\phi, q, p) \subset m^3(M, \psi, q, p)$ if and only if $\sup_{s,t,q} \frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$.

Theorem 3.9. The double space $m^3(M, \phi, q, p)$ is solid and symmetric.

Proof. Let $(x_{ijl}) \in m^3(M, \phi, q, p)$. Then

$$\sup_{s,t,q\geq 1, \sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty. \tag{3.3}$$

Let (λ_{ij}) be a double sequence of scalars with $|\lambda_{ijl}| \leq 1$ for all $i, j, l \in \mathbb{N}$. Then the result follows from (3.3) and the following inequality

$$\sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{\lambda_{ijl}x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} \leq \sum_{(i,j,l)\in\sigma} \left[|\lambda_{ijl}| M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} \quad \text{(by the Remark)}$$

$$\leq \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}}.$$

The symmetricity of the space follows from the definition of the triple space $m^3(M, \phi, q, p)$ and symmetric triple sequence space.

The following result follows from Theorem 3.9 and the Lemma.

Corollary 3.10. The triple space $m^3(M, \phi, q, p)$ is monotone.

The proof of the following result is a routine work.

Proposition 3.11. The triple spaces $l_p^3(M, q)$ and $l_\infty^3(M, q, p)$ are solid and as such are monotone.

Theorem 3.12. $l_p^3(M, q) \subset m^3(M, \phi, q, p) \subset l_\infty^3(M, q, p)$.

Proof. Let $(x_{ijl}) \in l_p^3(M, q)$. Then we have

$$\sum_{i,j,l=1,1,1}^{\infty} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty, \text{for some } r > 0. \tag{3.4}$$

Since $\{\phi_{n,m}\}$ is monotonic increasing, so we have

$$\frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} \leq \frac{1}{\phi_{1,1,1}} \sum_{(i,j,)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ij}l} < \infty.$$

Hence

$$\sup_{s,t,q\geq 1, \sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ij}l} < \infty.$$

Thus $(x_{ijl}) \in m^3(M, \phi, q, p)$. Therefore $l_p^3(M, q) \subset m^3(M, \phi, q, p)$. Next let $(x_{ijl}) \in m^3(M, \phi, q, p)$. Then we have

$$\sup_{s,t,q\geq 1, \sigma\in P_{stq}} \frac{1}{\phi_{s,t,q}} \sum_{(i,j,l)\in\sigma} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty, \text{forsome} r > 0.$$

So,

$$\sup_{i,j,l\in\mathbb{N}} \frac{1}{\phi_{1,1,1}} \left[M\left(q\left(\frac{x_{ijl}}{r}\right)\right)\right]^{p_{ijl}} < \infty, \text{for some } r > 0 \text{(on taking cardinality of } \sigma \text{ to be 1)}.$$

Therefore $(x_{ijl}) \in l_\infty^3(M, q, p)$. Hence $m^3(M, \phi, q, p) \subset l_\infty^3(M, q, p)$. This completes the proof.

Theorem 3.13.(i) $m^3(M,\phi,q,p) = l_p^3(M,q)$ if and only if $\sup_{s,t,q\geq 1}\phi_{s,t,q} < \infty$.

(ii) $m^3(M,\phi,q,p) = l_\infty^3(M,q,p)$ if and only if $\sup_{s,t,q\geq 1}\frac{stq}{\phi_{s,t,q}} < \infty$.

Proof.(i) It is clear that $m^3(M,\psi,q,p) = l_p^3(M,q)$ when $\psi_{s,t,q} = 1$ for all $s,t,q \in \mathbb{N}$. By Theorem 3.4., $m^3(M,\phi,q,p) \subset m^3(M,\psi,q,p)$ if and only if $\sup_{s,t}\frac{\phi_{s,t,q}}{\psi_{s,t,q}} < \infty$ i.e. $\sup_{s,t,q}\phi_{s,t,q} < \infty$. By Theorem 3.11., $m^3(M,\phi,q,p) = l_p^3(M,q)$ if and only if $\sup_{s,t,q}\phi_{s,t,q} < \infty$.

(ii) We have $m^3(M,\psi,q,p) = l_\infty^3(M,q)$ if $\psi_{s,t,q} = stq$ for all $s,t,q \in \mathbb{N}$. By Theorem 3.4. and Theorem 3.11., it follows that $m^3(M,\phi,q,p) = l_\infty^3(M,q,p)$ if and only if $\sup_{s,t,q\geq 1}\frac{stq}{\phi_{s,t,q}} < \infty$.

This completes the proof.

The proof of the following result is routine work.

Proposition 3.14. Let M be an Orlicz function, q_1 and q_2 be seminorms. Then

(i) $m^3(M,\phi,q_1,p) \cap m^3(M,\phi,q_2,p) \subset m^2(M,\phi,q_1+q_2,p)$,

(ii) If q_1 is stronger than q_2, then $m^3(M,\phi,q_1,p) \subset m^3(M,\phi,q_2,p)$,

(iii) $l_\infty^3(M,q_1,p) \cap l_\infty^3(M,q_2,p) \subset l_\infty^3(M,q_1+q_2,p)$,

(iv) If q_1 is stronger than q_2, then $l_\infty^3(M,q_1,p) \subset l_\infty^3(M,q_2,p)$,

(v) $l_p^3(M,q_1) \cap l_p^3(M,q_2) \subset l_p^3(M,q_1+q_2)$,

(vi) If q_1 is stronger than q_2, then $l_p^3(M,q_1) \subset l_p^3(M,q_2)$.

Particular Cases: If one considers a normed linear space $(X,\|.\|)$ instead of a seminormed space (X,q), then one will get $m^3(M,\phi,\|.\|,p)$, which will be a normed linear space, normed by

$$\|(x_{ij})\| = \inf\left\{r^{\frac{p_{ij}}{J}} > 0: \sup_{s,t,q\geq 1,\sigma\in P_{stq}}\frac{1}{\phi_{s,t,q}}\sum_{(i,j,l)\in\sigma} M\left(\left\|\frac{x_{ij}}{r}\right\|\right) \leq 1\right\}, \text{where} J = max(1,2^{H-1}).$$

The triple space $m^3(M,\phi,\|.\|,p)$ will be a solid, monotone and symmetric space. Further most of the results proved in the previous section will be true for this space too.

IV.CONCLUSION

The main result of this conference paper II is to introduce some properties of triple sequence spaces.

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