Tschirnhausen Helical Surface in Three Dimensional Euclidean Space

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Abstract – The main aim of this research is to reveal Tschirnhausen helical surface in three-dimensional Euclidean space $\mathbb{E}^3$. We construct Tschirnhausen helical surface, and obtain its Gauss map, Gaussian curvature, mean curvature. Moreover, we compute some relations of the curvatures of that kind surface.

Keywords – Euclidean 3-Space, Helical Surface, Tschirnhausen Helical Surface, Gauss Map, Curvature

I. Introduction

Mathematicians, especially geometers have been studied the surface theory for almost five hundred years. Some books can be seen about the theory in literature [1-7].

We investigate the Tschirnhausen helical surface in three dimensional Euclidean space $\mathbb{E}^3$.

In Section 1, we indicate some definitions of 3-space.

In Section 2, we serve the helical surface and then, we present Tschirnhausen helical surface, and compute its Gaussian and mean curvatures. We present some relations for the curvatures of the surface.

We serve a conclusion in Section 3.

In this paper, we equivalent a vector $(\mathrm{p}, \mathrm{q}, \mathrm{r})$ with its transpose.

Next, in $\mathbb{E}^3$, we define the fundamental forms *I*, *II*, shape operator matrix, Gauss curvature, mean curvature of the surface $\boldsymbol{x} = \boldsymbol{x}(u, v)$.

Let $\boldsymbol{x}$ be a surface M^2 in $\mathbb{E}^3$. The cross product of $\vec{m} = (m_1, m_2, m_3)$ and $\vec{n} = (n_1, n_2, n_3)$ of $\mathbb{E}^3$ is defined by

$$\vec{m} \times \vec{n} = \begin{vmatrix} e_1 & e_2 & e_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix},$$

where $|\quad|$ is described as determinant. We consider the following matrices

$$I = (g_{ij})_{2\times 2},$$

and

$$II = (h_{ij})_{2\times 2},$$

where

$$g_{11} = \boldsymbol{x}_u \cdot \boldsymbol{x}_u,$$

$$g_{12} = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = g_{21},$$

$$g_{22} = \boldsymbol{x}_v \cdot \boldsymbol{x}_v,$$

$$h_{11} = \boldsymbol{x}_{uu} \cdot G,$$

$$h_{12} = \boldsymbol{x}_{uv} \cdot G = h_{21},$$

$$h_{22} = \boldsymbol{x}_{vv} \cdot G,$$

" $\cdot$ " is a Euclidean dot product, the Gauss map of the surface is given by

$$G = G(u, v) = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}.$$

We calculate $I^{-1}.II$, and it follows the shape operator matrix

$$\boldsymbol{\mathcal{S}} = \frac{1}{detI}\begin{pmatrix} g_{22}h_{11} - g_{12}h_{12} & g_{22}h_{12} - g_{12}h_{22} \\ g_{11}h_{12} - g_{12}h_{11} & g_{11}h_{22} - g_{12}h_{12} \end{pmatrix}.$$

Finally, we obtain the following formula of Gaussian curvature

$$K = det(\boldsymbol{\mathcal{S}})$$

$$= \frac{h_{11}h_{22} - {h_{12}}^2}{g_{11}g_{22} - {g_{12}}^2}$$

and the mean curvature formula

$$H = \frac{1}{2}tr(\boldsymbol{\mathcal{S}})$$

$$= \frac{g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}}{2(g_{11}g_{22} - {g_{12}}^2)},$$

respectively. When $K = 0$, the surface $\boldsymbol{x}$ is flat, and it is minimal when $H = 0$.

II. Tschirnhausen Helical Surface

In this section, we present the surface of rotation and the helical surface in $\mathbb{E}^3$.

Consider open interval I, let $\gamma : \mathrm{I} \subset \mathbb{R} \longrightarrow \Pi$ be a curve, and ℓ be a line in Π.

We define the surface of rotation as a surface rotating the generating curve γ about the axis ℓ.

When the generating curve rotates about ℓ, it replaces parallel lines orthogonal to ℓ, then the accelerate of replacement is in proportion to the accelerate of rotation.

Therefore, the above surface is named the *helical surface* having axis ℓ, pitch $\wp \in \mathbb{R}^+$.

The orthogonal matrix is given by

$$\mathfrak{O}(v) = \begin{pmatrix} cosv & -sinv & 0 \\ sinv & cosv & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here, $v \in \mathbb{R}$. Therefore, $\mathfrak{O}$ supplies the following, simultaneously,

$$\mathfrak{O}.\ell = \ell, \quad \mathfrak{O}^t.\mathfrak{O} = \mathfrak{O}.\mathfrak{O}^t = \mathfrak{I}_3, \ \det \mathfrak{O} = 1,$$

where $\mathfrak{I}_3$ is the identity matrix of order 3.

If the rotation axis be ℓ, there is a transformation transformed ℓ to the axis x_3.

The generating curve is given by

$$\gamma(u) = (\mathfrak{f}(u), 0, \mathfrak{g}(u)),$$

where $\mathfrak{f}(u), \mathcal{g}(u) \in C^k(\mathrm{I}, \mathbb{R})$.

Hence, the helical surface spanned by the $(0,0,1)$ having pitch $\wp$, is defined by

$$\mathcal{H}(u,v) = \mathfrak{O}(v).\gamma(u) + \wp\, v\, \ell^t,$$

where $u \in \mathrm{I}$, $v \in [0, 2\pi)$.

So, we have the following helical surface

$$\mathcal{H}(u,v) = \begin{pmatrix} \mathfrak{f}(u) cosv \\ \mathfrak{f}(u) sinv \\ \mathfrak{g}(u) + \wp v \end{pmatrix}.$$

When $\wp = 0$, the helical surface is transform to the surface of rotation.

Considering the following Tchirnhausen curve for $a, b, h \in \mathbb{R}$ in $\mathbb{E}^3$

$$\gamma[n, \mathrm{a}](u) = \big(\mathcal{P}(u), 0, \mathcal{Q}(u)\big)$$

$$= \mathrm{a}\begin{pmatrix} \dfrac{\cos u}{\cos^n\left(\frac{u}{3}\right)} \\ 0 \\ \dfrac{\sin u}{\cos^n\left(\frac{u}{3}\right)} \end{pmatrix},$$

we calculate the Gauss map, and also find the curvatures of the Tchirnhausen surface. See [3] for details of Tschirnhausen curve.

In $\mathbb{E}^3$, the Tschirnhausen helical surface (see Figure 1 and Figure 2 for $\wp = 1$, see Figure 3 and Figure 4 for $\wp = 0$) spanned by the $(0,0,1)$, has pitch $\wp \in \mathbb{R}^+$, taking $a = 1$ on generating curve, is defined by

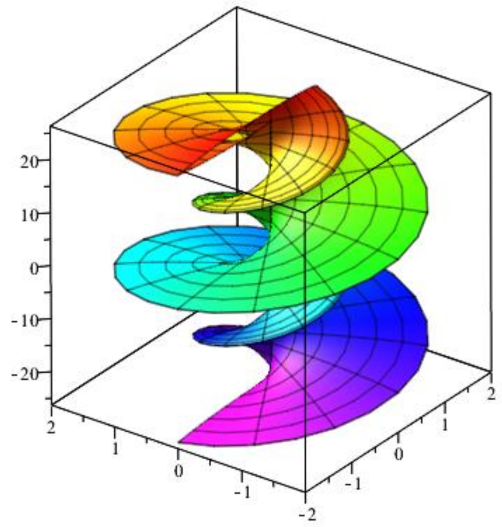

Fig. 1: Tschirnhausen helical surface, $\wp = 1$

$$\boldsymbol{\mathcal{T}}(u,v) = \begin{pmatrix} \dfrac{\cos u \cos v}{\cos\left(\frac{u}{3}\right)} \\ \dfrac{\cos u \sin v}{\cos\left(\frac{u}{3}\right)} \\ \dfrac{\sin u}{\cos\left(\frac{u}{3}\right)} + \wp v \end{pmatrix},$$

where the generating space curve is described by

$$\gamma(u) = \left(\frac{\cos u}{\cos\left(\frac{u}{3}\right)}, 0, \frac{\sin u}{\cos\left(\frac{u}{3}\right)}\right),$$

and $u \in \mathrm{I}$, $\wp \in \mathbb{R}^+$, $v \in [0, 2\pi)$.

By using the first differentials of the Tschirnhausen helical surface $\boldsymbol{\mathcal{T}}(u,v)$ depends on u and v, we reveal the following first quantities

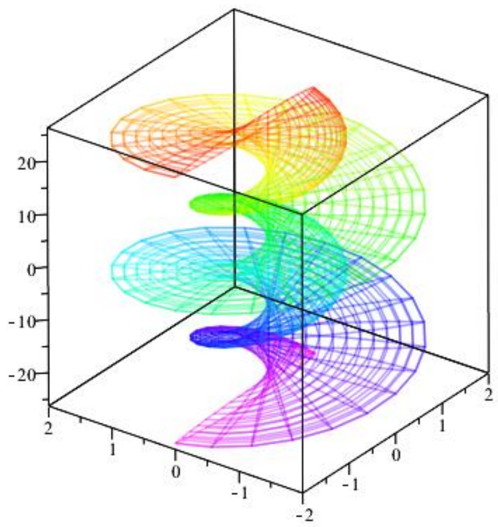

Fig. 2: Curves on Tschirnhausen helical surface, $\wp = 1$

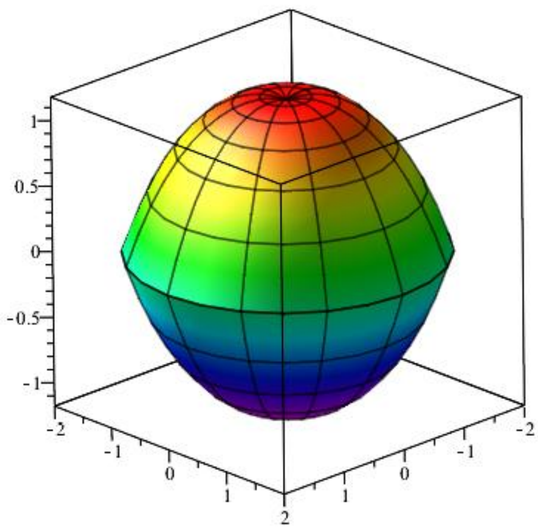

Fig. 3: Tschirnhausen rotational surface, $\wp = 0$

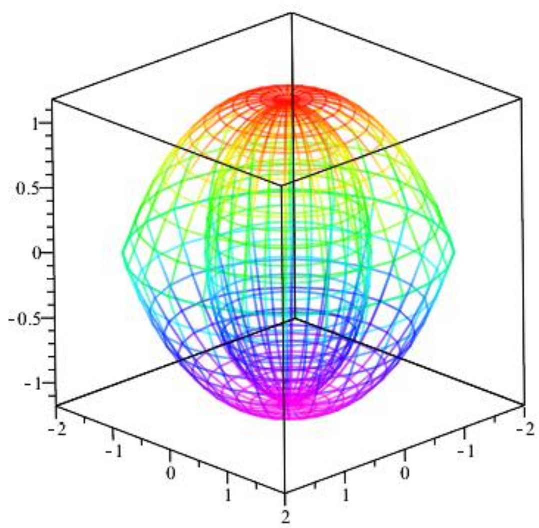

Fig. 4: Curves on Tschirnhausen rotational surface, $\wp = 0$

$$g_{11} = \frac{8\cos^2\left(\frac{u}{3}\right) + 1}{9\cos^4\left(\frac{u}{3}\right)},$$

$$g_{12} = \wp\,\frac{\left(8\cos^4\left(\frac{u}{3}\right) - 4\cos^2\left(\frac{u}{3}\right) - 1\right)}{3\cos^2\left(\frac{u}{3}\right)},$$

$$g_{22} = \wp^2 + 16\cos^4\left(\frac{u}{3}\right) - 24\cos^2\left(\frac{u}{3}\right) + 9.$$

By using above results, we get the following

$$\begin{aligned}\det(g_{ij}) = \det I \\ &= \Bigg(9 - 64\wp^2\cos^8\left(\frac{u}{3}\right) \\ &\quad + (64\,\wp^2 + 128)\cos^6\left(\frac{u}{3}\right) \\ &\quad - 176\cos^4\left(\frac{u}{3}\right) + 48\cos^2\left(\frac{u}{3}\right)\Bigg) \\ &\quad / \, 9\cos^4\left(\frac{u}{3}\right).\end{aligned}$$

Hence, the Gauss map (see Figure 3 and Figure 4) of Tschirnhausen helical surface is given by

$$G(u,v) = \frac{1}{W^{1/2}}\begin{pmatrix} G_1(u,v) \\ G_2(u,v) \\ G_3(u,v)\end{pmatrix},$$

where

$$\begin{aligned}G_1 = 3\wp\sin u\cos^2\left(\frac{u}{3}\right)\sin v \\ &\quad - \wp\cos\left(\frac{u}{3}\right)\cos u\sin\left(\frac{u}{3}\right)\sin v \\ &\quad + \sin u\cos v\cos u\sin\left(\frac{u}{3}\right) \\ &\quad + 3\cos\left(\frac{u}{3}\right)\cos v\cos^2 u,\end{aligned}$$

$$G_2 = 3\wp \sin u \cos^2\left(\frac{u}{3}\right)\cos v - \wp \cos\left(\frac{u}{3}\right)\cos u \sin\left(\frac{u}{3}\right)\cos v - \sin u \sin v \cos u \sin\left(\frac{u}{3}\right) - 3\cos\left(\frac{u}{3}\right)\sin v \cos^2 u,$$

$$G_3 = -8\sin\left(\frac{u}{3}\right)\cos^4\left(\frac{u}{3}\right)\left(4\cos^2\left(\frac{u}{3}\right) - 3\right),$$

and

$$W = \left(9 - 64\wp^2\cos^8\left(\frac{u}{3}\right) + (64\,\wp^2 + 128)\cos^6\left(\frac{u}{3}\right) - 176\cos^4\left(\frac{u}{3}\right) + 48\cos^2\left(\frac{u}{3}\right)\right) / \cos\left(\frac{u}{3}\right).$$

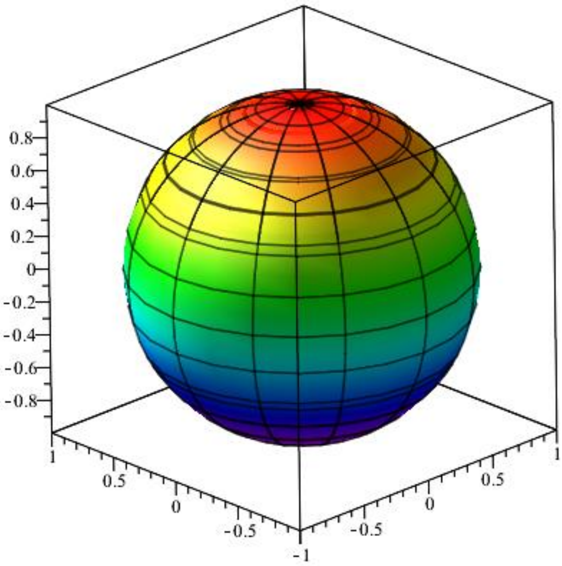

Fig. 5: Gauss map of Tschirnhausen helical surface, $\wp = 1$

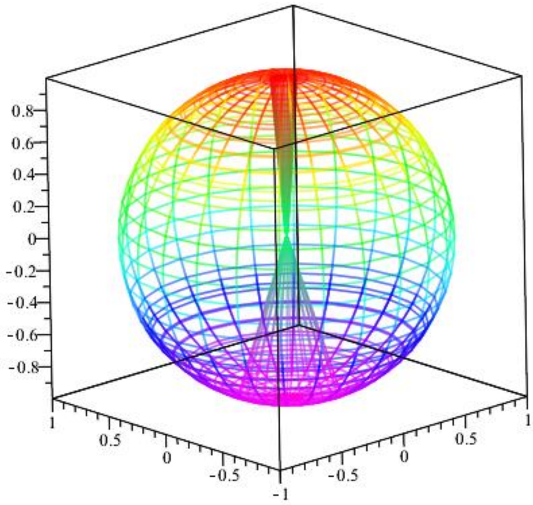

Fig. 6: Curves on Gauss map of Tschirnhausen helical surface, $\wp = 1$

Therefore, the mean curvature of Tschirnhausen helical surface $\boldsymbol{\mathcal{T}}(u, v)$ is described by

$$H = \frac{\hbar(u)}{W^{3/2}},$$

where

$$
\begin{aligned}
\mathscr{h}(u) = 1/2 * (&3456 \cos(u) * \sin(u) * \sin(v) * \cos(v) * \cos(1/3 * u)^{11} * \mathscr{p} + (1152 * \sin(u) \\
&* \cos(v) * \sin(v) * \cos(u)^{2} * \mathscr{p} - 2688 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u)^{2} \\
&* \mathscr{p} + 3456 * \cos(v)^{2} * \cos(u)^{3} + 1728 * \cos(v) * \sin(v) * \sin(1/3 * u) * \mathscr{p} - 576 \\
&* \cos(v)^{2} * \cos(u) - 1152 * \cos(u)) * \cos(1/3 * u)^{10} + (-896 * \cos(v) * \sin(v) \\
&* \sin(1/3 * u) * \cos(u)^{3} * \mathscr{p} + 1152 * \cos(u)^{4} * \cos(v)^{2} + 2688 * \cos(u)^{2} \\
&* \cos(v)^{2} * \sin(u) * \sin(1/3 * u) - 6336 * \sin(u) * \cos(v) * \sin(v) * \cos(u) * \mathscr{p} \\
&+ 576 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u) * \mathscr{p} - 192 * \cos(u)^{2} * \cos(v)^{2} \\
&- 448 * \cos(v)^{2} * \mathscr{p}^{2} - 384 * \cos(u)^{2} + 224 * \mathscr{p}^{2}) * \cos(1/3 * u)^{9} + (896 \\
&* \cos(u)^{3} * \sin(u) * \cos(v)^{2} * \sin(1/3 * u) + 128 * \cos(v)^{2} * \cos(u)^{3} * \mathscr{p}^{2} \\
&- 1600 * \sin(u) * \cos(v) * \sin(v) * \cos(u)^{2} * \mathscr{p} + 4224 * \cos(v) * \sin(v) * \sin(1/3 \\
&* u) * \cos(u)^{2} * \mathscr{p} + 256 * \cos(v)^{2} * \cos(5/3 * u) * \mathscr{p}^{2} + 160 * \cos(v)^{2} \\
&* \cos(7/3 * u) * \mathscr{p}^{2} - 6336 * \cos(v)^{2} * \cos(u)^{3} - 64 * \cos(u)^{3} * \mathscr{p}^{2} - 96 \\
&* \cos(v)^{2} * \cos(u) * \mathscr{p}^{2} + 32 * \sin(v) * \cos(v) * \sin(11/3 * u) * \mathscr{p} + 32 * \sin(v) \\
&* \cos(v) * \sin(5/3 * u) * \mathscr{p} + 128 * \sin(v) * \cos(v) * \sin(7/3 * u) * \mathscr{p} - 2464 * \cos(v) \\
&* \sin(v) * \sin(1/3 * u) * \mathscr{p} - 128 * \mathscr{p}^{2} * \cos(5/3 * u) - 80 * \mathscr{p}^{2} * \cos(7/3 * u) \\
&+ 1440 * \cos(v)^{2} * \cos(u) + 48 * \cos(u) * \mathscr{p}^{2} + 1728 * \cos(u)) * \cos(1/3 * u)^{8} \\
&+ (1408 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u)^{3} * \mathscr{p} + 216 * \sin(u) * \cos(v) \\
&* \sin(v) * \cos(u) * \mathscr{p}^{3} - 2112 * \cos(u)^{4} * \cos(v)^{2} - 4224 * \cos(u)^{2} * \cos(v)^{2} \\
&* \sin(u) * \sin(1/3 * u) + 3672 * \sin(u) * \cos(v) * \sin(v) * \cos(u) * \mathscr{p} - 864 * \cos(v) \\
&* \sin(v) * \sin(1/3 * u) * \cos(u) * \mathscr{p} + 480 * \cos(u)^{2} * \cos(v)^{2} + 224 * \cos(v)^{2} \\
&* \mathscr{p}^{2} + 576 * \cos(u)^{2} - 112 * \mathscr{p}^{2}) * \cos(1/3 * u)^{7} + (72 * \cos(u)^{2} * \sin(u) \\
&* \sin(v) * \cos(v) * \mathscr{p}^{3} - 168 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u)^{2} * \mathscr{p}^{3} \\
&- 1408 * \cos(u)^{3} * \sin(u) * \cos(v)^{2} * \sin(1/3 * u) + 152 * \cos(v)^{2} * \cos(u)^{3} \\
&* \mathscr{p}^{2} + 968 * \sin(u) * \cos(v) * \sin(v) * \cos(u)^{2} * \mathscr{p} - 1800 * \cos(v) * \sin(v) \\
&* \sin(1/3 * u) * \cos(u)^{2} * \mathscr{p} + 108 * \sin(v) * \cos(v) * \sin(1/3 * u) * \mathscr{p}^{3} - 128 \\
&* \cos(v)^{2} * \cos(5/3 * u) * \mathscr{p}^{2} - 80 * \cos(v)^{2} * \cos(7/3 * u) * \mathscr{p}^{2} + 3672 \\
&* \cos(v)^{2} * \cos(u)^{3} + 32 * \cos(u)^{3} * \mathscr{p}^{2} + 12 * \cos(v)^{2} * \cos(u) * \mathscr{p}^{2} - 16 \\
&* \sin(v) * \cos(v) * \sin(11/3 * u) * \mathscr{p} - 16 * \sin(v) * \cos(v) * \sin(5/3 * u) * \mathscr{p} - 64 \\
&* \sin(v) * \cos(v) * \sin(7/3 * u) * \mathscr{p} + 908 * \cos(v) * \sin(v) * \sin(1/3 * u) * \mathscr{p} + 64 \\
&* \mathscr{p}^{2} * \cos(5/3 * u) + 40 * \mathscr{p}^{2} * \cos(7/3 * u) - 1188 * \cos(v)^{2} * \cos(u) - 96 \\
&* \cos(u) * \mathscr{p}^{2} - 648 * \cos(u)) * \cos(1/3 * u)^{6} + (-56 * \cos(u)^{3} * \sin(v) * \cos(v) \\
&* \sin(1/3 * u) * \mathscr{p}^{3} + 72 * \cos(u)^{4} * \cos(v)^{2} * \mathscr{p}^{2} + 168 * \sin(u) * \cos(v)^{2} \\
&* \sin(1/3 * u) * \cos(u)^{2} * \mathscr{p}^{2} - 600 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u)^{3} * \mathscr{p}
\end{aligned}
$$

$- 72 * \sin(u) * \cos(v) * \sin(v) * \cos(u) * p^3 + 36 * \cos(u) * \sin(v) * \cos(v)$
$* \sin(1/3 * u) * p^3 + 1224 * \cos(u)^4 * \cos(v)^2 + 1800 * \cos(u)^2 * \cos(v)^2$
$* \sin(u) * \sin(1/3 * u) - 12 * \cos(v)^2 * \cos(u)^2 * p^2 - 504 * \sin(u) * \cos(v)$
$* \sin(v) * \cos(u) * p + 324 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u) * p - 396$
$* \cos(u)^2 * \cos(v)^2 - 24 * p^2 * \cos(u)^2 + 56 * \cos(v)^2 * p^2 - 216$
$* \cos(u)^2 - 28 * p^2) * \cos(1/3 * u)^5 + (56 * \cos(u)^3 * \sin(u) * \cos(v)^2$
$* \sin(1/3 * u) * p^2 - 24 * \cos(u)^2 * \sin(u) * \sin(v) * \cos(v) * p^3 + 12 * \cos(v)$
$* \sin(v) * \sin(1/3 * u) * \cos(u)^2 * p^3 + 600 * \cos(u)^3 * \sin(u) * \cos(v)^2$
$* \sin(1/3 * u) - 88 * \cos(v)^2 * \cos(u)^3 * p^2 - 232 * \sin(u) * \cos(v) * \sin(v)$
$* \cos(u)^2 * p + 60 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u)^2 * p - 32 * \cos(v)^2$
$* \cos(5/3 * u) * p^2 - 20 * \cos(v)^2 * \cos(7/3 * u) * p^2 - 504 * \cos(v)^2$
$* \cos(u)^3 + 8 * \cos(u)^3 * p^2 + 48 * \cos(v)^2 * \cos(u) * p^2 - 4 * \sin(v) * \cos(v)$
$* \sin(11/3 * u) * p - 4 * \sin(v) * \cos(v) * \sin(5/3 * u) * p - 16 * \sin(v) * \cos(v)$
$* \sin(7/3 * u) * p - 16 * \cos(v) * \sin(v) * \sin(1/3 * u) * p + 16 * p^2 * \cos(5/3 * u)$
$+ 10 * p^2 * \cos(7/3 * u) - 72 * \cos(u)^3 + 324 * \cos(v)^2 * \cos(u) - 6 * \cos(u)$
$* p^2) * \cos(1/3 * u)^4 + (4 * \cos(u)^3 * \sin(v) * \cos(v) * \sin(1/3 * u) * p^3 - 24$
$* \cos(u)^4 * \cos(v)^2 * p^2 - 12 * \sin(u) * \cos(v)^2 * \sin(1/3 * u) * \cos(u)^2 * p^2$
$+ 20 * \cos(v) * \sin(v) * \sin(1/3 * u) * \cos(u)^3 * p - 168 * \cos(u)^4 * \cos(v)^2$
$- 60 * \cos(u)^2 * \cos(v)^2 * \sin(u) * \sin(1/3 * u) + 12 * \cos(v)^2 * \cos(u)^2 * p^2$
$+ 18 * \sin(u) * \cos(v) * \sin(v) * \cos(u) * p - 24 * \cos(u)^4 + 108 * \cos(u)^2$
$* \cos(v)^2 - 24 * \cos(u)^2 * \sin(u) * \sin(1/3 * u)) * \cos(1/3 * u)^3 + (-4$
$* \cos(u)^3 * \sin(u) * \cos(v)^2 * \sin(1/3 * u) * p^2 - 20 * \cos(u)^3 * \sin(u)$
$* \cos(v)^2 * \sin(1/3 * u) + 6 * \sin(u) * \cos(v) * \sin(v) * \cos(u)^2 * p - 6 * \cos(v)$
$* \sin(v) * \sin(1/3 * u) * \cos(u)^2 * p + 18 * \cos(v)^2 * \cos(u)^3 - 8 * \cos(u)^3$
$* \sin(u) * \sin(1/3 * u) - 9 * \cos(u)^3) * \cos(1/3 * u)^2 + (-2 * \cos(v) * \sin(v)$
$* \sin(1/3 * u) * \cos(u)^3 * p + 6 * \cos(u)^4 * \cos(v)^2 + 6 * \cos(u)^2 * \cos(v)^2$
$* \sin(u) * \sin(1/3 * u) - 3 * \cos(u)^4 - 3 * \cos(u)^2 * \sin(u) * \sin(1/3 * u))$
$* \cos(1/3 * u) + 2 * \cos(u)^3 * \sin(u) * \cos(v)^2 * \sin(1/3 * u) - \cos(u)^3 * \sin(u)$
$* \sin(1/3 * u))/(-3 * \cos(1/3 * u) - \cos(u)).$

Therefore, the Gaussian curvature of Tschirnhausen helical surface $\mathcal{T}(u, v)$ is given by

$$K = \frac{k(u)}{W^2}.$$

$$
\begin{aligned}
k(u) = 1/2048 * (((&-5984 * p^2 + 119392) * \cos(v)^4 + (5984 * p^2 - 123482) * \cos(v)^2 \\
&+ 2584 * p^2 + 28896) * \cos(4/3 * u) + ((-48 * p^2 - 16) * \cos(v)^4 + (48 * p^2 \\
&+ 1286) * \cos(v)^2 - 116 * p^2 + 132) * \cos(20/3 * u) + ((-4 * p^2 - 136) \\
&* \cos(v)^4 + (4 * p^2 + 680) * \cos(v)^2 - 16 * p^2 + 12) * \cos(22/3 * u) \\
&+ ((45696 * p^2 + 71464) * \cos(v)^4 + (-45696 * p^2 - 71362) * \cos(v)^2 \\
&+ 8721 * p^2 + 18144) * \cos(8/3 * u) + (-5120 * \cos(u)^8 + (-7040 * p^2 \\
&+ 54656) * \cos(u)^6 + (91584 * p^2 + 143424) * \cos(u)^4 + (-5832 * p^2 \\
&+ 3888) * \cos(u)^2 - 78732 * p^2) * \cos(v)^4 + (-15616 * p^2 + 20736) \\
&* \cos(u)^6 + (12096 * p^2 + 28512) * \cos(u)^4 + (24832 * \cos(u)^8 + (7040 \\
&* p^2 - 26560) * \cos(u)^6 + (-91584 * p^2 - 191376) * \cos(u)^4 + (5832 * p^2 \\
&- 3888) * \cos(u)^2 + 78732 * p^2) * \cos(v)^2 + ((27948 * p^2 + 46592) \\
&* \cos(v)^4 + (-27948 * p^2 - 44084) * \cos(v)^2 + 2736 * p^2 + 12204) \\
&* \cos(10/3 * u) + ((-288 * p^2 + 5648) * \cos(v)^4 + (288 * p^2 - 2860) \\
&* \cos(v)^2 - 1346 * p^2 + 1932) * \cos(16/3 * u) + (-4 * \cos(v)^4 + 30 \\
&* \cos(v)^2) * \cos(26/3 * u) + ((-68952 * p^2 + 136168) * \cos(v)^4 + (68952 \\
&* p^2 - 142532) * \cos(v)^2 - 10336 * p^2 + 32736) * \cos(2/3 * u) + ((1652 \\
&* p^2 + 13868) * \cos(v)^4 + (-1652 * p^2 - 10510) * \cos(v)^2 - 2224 * p^2 \\
&+ 4152) * \cos(14/3 * u) + 2 * \cos(28/3 * u) * \cos(v)^2 - 128 * \cos(u)^8 * p^2 + 2 \\
&* p * \sin(26/3 * u) * \sin(v) * \cos(v) - 1024 * \sin(v) * \cos(u) * (\cos(u)^6 + 27/4 \\
&* \cos(u)^4 - 2511/8 * \cos(u)^2 - 2187/64) * p * \sin(u) * \cos(v)^3 + 4096 \\
&* \sin(v) * \cos(u) * p * (\cos(u)^6 + 813/64 * \cos(u)^4 - 3321/64 * \cos(u)^2 \\
&- 2187/512) * \sin(u) * \cos(v) + 52528 * (\cos(v)^2 - 7743/13132) * \sin(v) * p \\
&* \cos(v) * \sin(2/3 * u) + 67896 * \sin(v) * (\cos(v)^2 - 223/414) * p * \cos(v) \\
&* \sin(10/3 * u) + 15400 * \sin(v) * p * (\cos(v)^2 - 391/1100) * \cos(v) * \sin(14/3 \\
&* u) + 85888 * \sin(v) * (\cos(v)^2 - 382/671) * p * \cos(v) * \sin(4/3 * u) + 89496 \\
&* (\cos(v)^2 - 6100/11187) * \sin(v) * p * \cos(v) * \sin(8/3 * u) + 3136 * \sin(v) * p \\
&* \cos(v) * (\cos(v)^2 + 33/98) * \sin(16/3 * u) - 352 * \sin(v) * (\cos(v)^2 - 27/11) \\
&* p * \cos(v) * \sin(20/3 * u) - 88 * \sin(v) * (\cos(v)^2 - 28/11) * p * \cos(v) \\
&* \sin(22/3 * u) - 19683 * p^2 + 23328 * p^2 * \cos(u)^2)/(p^2 * \cos(2/3 * u)^4 \\
&+ (2 * p^2 - 4) * \cos(2/3 * u)^3 - \cos(2/3 * u)^2 + (-2 * p^2 + 4) * \cos(2/3 * u) \\
&- p^2 - 5/4)^2/(\cos(2/3 * u) + 1)^4.
\end{aligned}
$$

III. CONCLUSION

By using above findings, we have the following results.

Corollary 1. Let $\mathcal{T} : \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ be an immersion defined by $\mathcal{T}(u,v)$. Then, M^2 has the following relation

$$H^2 = \frac{\hbar^2}{4\mathcal{k}} K.$$

Corollary 2. Let $\mathcal{T} : \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ be an immersion obtained by $\mathcal{T}(u,v)$. M^2 has umbilic point if and only if the following relation holds

$$\hbar = \mp 2\mathcal{k}^{1/2}.$$

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