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C-Supplemented Modules

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Abstract – In this paper, we present the c-supplemented modules and give the fundamental properties of c-supplemented modules. For any arbitrary ring we define a module N as *c-supplemented* if, for each submodule K of N, there is a submodule L in N such that N=K+L and K ∩ L is crumbling. In particular, we demonstrate that the league of c-supplemented modules exhibits closure properties under various operations, including direct sums, submodules, homomorphic images and sums. Furthermore, we confirm that for a ring S, $_SS$ is c-supplemented if every left S-module is c-supplemented.

Keywords – Supplemented, C-Supplemented, Semisimple, SSI-Ring, Crumble

I. Introduction

In the intricate and captivating realm of module theory, we encounter a symphony of notations and concepts that serve as the foundation for our understanding of modules. In the course of this investigation involving a module N, we employ specific notations to facilitate our analysis. Specifically we use the symbols Rad(N), E(N) and Soc(N) to represent the radical, injective hull and socle of N, respectively. Furthermore we recall the concept of the crumbling submodule of N, which is defined as the aggregate of all submodules within N that exhibit the property of crumbling. This submodule is denoted by C(N) as in [11]. We use the notation Y ≤ N to indicate that Y is a submodule of N. A submodule K of a module N is considered *small* in N and is demonstrated as K ≪ N if, for every proper submodule S of N, it holds that N ≠ K+S (see [3]). Let Y be an R-module. A submodule X of Y is called *essential* in Y which denoted by X ⊴ Y, if X ∩ K≠ 0 for every nonzero K≤N (see [10]). One of the fundamental concepts in module theory is that of a semisimple module, often referred to as a module N being simple. The key characteristic of a semisimple module it's ability to decompose into a direct sum of submodules. In other words, a module N is semisimple if its submodules are direct summand. Another concept closely related to the decomposition of modules is that of a crumbling module. A module N is said to be *crumbling* if whenever $K \leq N$ there is a submodule $\frac{X}{K} \leq \frac{N}{K}$ such that $Soc(\frac{N}{K}) \oplus \frac{X}{K} = \frac{N}{K}$ (see [12]). This concept is crucial for understanding the decomposition of modules and their interactions at a submodule level.

It's well known that a module N is *supplemented* if its all submodules have a supplement in N. This concept is essential for understanding the complementarity and decomposition of modules into smaller, independent parts. In order to obtain fundamental knowledge, one may refer to [4-5] and to obtain knowledge about supplement types you can consult [1-3, 6-7, 10, 12].

Before we embark on our intellectual journey through the fascinating world of "C-supplemented modules," it is imperative to lay the foundation by clearly defining what these modules are. At the heart of this study lies the definition of c-supplemented modules, which is: for any arbitrary ring we define a module N as c-supplemented if, for every submodule K of N, there is a submodule L in N such that N=K+L and K ∩ L is crumbling.

II. Materials and Method

In this study, we will use deductive reasoning and mathematical proofs to establish and demonstrate key properties of c-supplemented modules. Our approach will involve presenting the definitions, theorems and proofs in a logical sequence, allowing for a systematic exploration of the subject matter. Through a comprehensive analysis of c-supplemented modules and their properties, we aim to provide a clear and insightful understanding of this intriguing concept within module theory.

III. Results

Theorem 2.1. A module *X* is c-supplemented if and only if $\frac{X}{C(X)}$ is semisimple.

Proof. (⇒) Let X be a c-supplemented module and $\frac{U}{C(X)} \leq \frac{X}{C(X)}$. Therefore C(X) ≤ U ≤ X. Since *X* is c-supplemented, there is a submodule *Y* of *X* such that X = U+Y and U ∩ Y is crumbling. Therefore U ∩ Y =C (U ∩ Y) and so U ∩ Y ⊆ C(X) by [11, Proposition 2-(2)]. It follows that $\frac{X}{C(X)} = \frac{U}{C(X)} + \frac{Y+C(X)}{C(X)}$. Notice that

$$\frac{U}{C(X)} \cap \left(\frac{Y+C(X)}{C(X)}\right) = \frac{U \cap Y+C(X)}{C(X)} = \frac{C(X)}{C(X)} = 0.$$

Hence $\frac{X}{C(X)}$ is semisimple.

(⇐) Let U ≤ X. By the hypothesis, we can write $\frac{X}{C(X)} = \frac{U+C(X)}{C(X)} \oplus \frac{Y}{C(X)}$ for some submodule $C(X) \leq Y \leq X$. Then X=U+C(X)+Y=U+Y. Since $\frac{U+C(X)}{C(X)} \cap \frac{Y}{C(X)} = \frac{U \cap Y+C(X)}{C(X)} = 0$, we obtain that $U \cap Y \subseteq C(X)$. So $U \cap Y$ is crumbling. Thus *X* is c-supplemented. ■

The following outcome is a direct aftermath of the above Theorem.

Corollary 2.1 If *X* is a c-supplemented module, then $C(X) \trianglelefteq X$.

Proof. Suppose that C(X) ∩ L=0 for some submodule *L* of *X*. It pursues that L≅$\frac{L \oplus C(X)}{C(X)}$. Since $\frac{X}{C(X)}$ is semisimple, we obtain that $\frac{L \oplus C(X)}{C(X)}$ is semisimple as a submodule of $\frac{X}{C(X)}$. Thus *L* is semisimple and then L ⊆ C(X). Hence C(X) ∩ L = L =0 ,as required. ■

Theorem 2.2. Suppose that X is a c-supplemented module. If so, each factor module of X is c-supplemented.

Proof. Let Y ≤ N≤ X. Since X is c-supplemented, N has a c-supplement, say, *K*, in *X*. Hence we are able to write X= N+ K and N ∩ K is crumbling. It pursues that $\frac{X}{Y} = \frac{N}{Y} + \frac{V+Y}{Y}$. Consider the natural homomorphism Ψ :X →$\frac{X}{Y}$. Now

Ψ (N∩K)= $\frac{N \cap K+Y}{Y}$ =$\frac{N \cap (K+Y)}{Y} = \frac{N}{Y} \cap \frac{K+Y}{Y}$ and so $\frac{N}{Y} \cap \frac{K+Y}{Y}$ is crumbling by [11, Proposition 2-(1)]. It means that $\frac{N}{Y}$ has a c-supplement in $\frac{X}{X}$. Hence $\frac{X}{Y}$ is c-supplemented. ■

Using Theorem 2.2 we have the subsequent case.

Corollary 2.2 All homomorphic image of a c-supplemented module is c-supplemented.

Proposition 2.1. Let X be a c-supplemented module. Then each submodule of X is c-supplemented.

Proof. Let Y be a submodule of *X*. Assume that K is any submodule of Y. Since X is c-supplemented, we can write the sum K+ L is N and K ∩ L is crumbling for some a submodule *L* of X. By the moduler law, we obtain that Y= Y ∩ X = Y ∩ (K+L) = Y ∩ L +K and then Y= K + Y ∩ L. In that case since K ∩ (Y ∩ L) = K ∩ Y ∩ L = K ∩ L is crumbling, It means that Y ∩ L is c-supplement of K in Y. Thus Y is c-supplemented. ■

Now we proceed to demonstrate that the direct sum of c-supplemented modules retains the possession of being c-supplemented.

Theorem 2.3. Assume that $\{M_i\}_{i\in I}$ is a family of c-supplemented modules and M= $\underset{i\in I}{\oplus}M_i$. Then M is c-supplemented.

Proof. It pursues from [11, Proposition 2-(5)] that $C(M)=\oplus_{i\in I}C(M_i)$ and so $\frac{M}{C(M)}=\frac{\underset{i\in I}{\oplus}M_i}{\oplus C(M_i)}\cong \underset{i\in I}{\oplus}\left(\frac{M_i}{C(M_i)}\right)$. By Theorem 2.1, we get that $\frac{M}{C(M)}$ is semisimple. Again applying Theorem 2.1, we obtain that M is c-supplemented. . ■

Corollary 2.3 Let M be a module and $\{M_i\}_{i\in I}$ be a family of c-supplemented submodules M_i of M. Then $\sum_{i\in I}M_i$ is c-supplemented.

Proof. Let N= $\underset{i\in I}{\oplus}M_i$. It pursues from Theorem 2.3 N is c-supplemented. Now we consider the homomorphism

Ψ : N $\to \sum_{i\in I}M_i$ by $\psi\ ((a_i)_{i\in I})=\sum_{i\in I_0}a_i$, where $I_0=\{i\in I|a_i\ \neq 0\}$. Therefore ψ is an epimorphism. By Corollary 2.2, we obtain that $\sum_{i\in I}M_i$ is c-supplemented. . ■

Now, we aim to provide a characterization of rings for which their modules exhibit c-supplemented properties. Also following theorem shedding light on the connection between c-supplemented rings and modules

Theorem 2.4. Let S be a ring. Then $_SS$ is c-supplemented if and only if every S module is c-supplemented.

Proof. (⇒) Let $_SS$ be c-supplemented and N be an arbitrary left S-module. Therefore there is an index set such that ψ :$S^I\to$ N is an epimorphism. Since $_SS$ is c-supplemented, it gets from Theorem 2.3 that S^I is c-supplemented. Hence *N* is c-supplemented according to Corollary 2.2.

(⇐) It is clear. . ■

Consider a ring R. It's well-established fact that R falls under the category of semisimple artinian rings if, and only if, each left R-module demonstrates semisimple characteristics. This equivalence further extends to conditions where every left R-module exhibits injective properties, acts as a projective module, and even extends to scenarios where every (semi)simple left R-module is also projective.It follows from [11, Theorem 3] that a ring *R* qualifies as a left noetherian V-ring if and only it meets the criteria of being an SSI-ring, which is further equivalent to the condition that all left *R*-module is crumbling.

Applying this case and Theorem 2.4, we derive the subsequent case:

Corollary 2.4 Let *S* be a SSI-ring. Then each left *S*-module is c-supplemented. . ■

Now we will provide an example showing that the reverse of Corollary 2.4 is not valid. For this we demand the subsequent cases.

Let's revisit the definition of a small module: a module *Y* is considered small module if it acts as a submodule contained within the injective hull of *E(X)* of X.

Lemma 2.1 Let S be a ring and *X* be a small *S*-module. Assume that $_SS$ is c-supplemented. Then X is crumbling.

Proof. By the assumption and Theorem 2.4, *E(X)* is c-supplemented, where *E(X)* is the injective hull of *X*. Since X is a small module, X + E(X) = E(X) and X ∩ E(X) =X is crumbling. This completes the proof. . ■

Proposition 2.2 Let S be a ring and *L* be an *S*-module. Assume that $_SS$ is c-supplemented. In that case Rad (L) is crumbling.

Proof. Let *Y* be any small submodule of *L*. It follows from Lemma 2.1 that *Y* is a crumbling module. By reason of *Rad(L)* is the sum of all small submodules of *L*, by [11, Proposition 2-(5)], we deduce that *Rad(L)* is crumbling. . ■

The next result is interesting.

Corollary 2.5 Let *S* be a ring. Suppose that $_SS$ is c-supplemented. Then *Rad(S)* is crumbling.

Proof. It pursues from Proposition 2.2. ■

Let S be a ring. In [13], *S* is termed as *semi-local* if $\frac{S}{Rad(S)}$ is semi-simple.

Now we give a class of c-supplemented rings.

Theorem 2.5 Let *S* be a semi-local ring with crumbling radical. Then $_SS$ is c-supplemented.

Proof. Since *Rad(S)* is crumbling, we have Rad(S)≤ C($_SS$). It pursues from the hypothesis that $\frac{S}{Rad(S)}$ is semisimple so $\frac{S}{C(\,_SS)}$ is a semisimple *S*-

module. Applying Theorem 2.1, we obtain that ${}_S S$ is c-supplemented. . ■

Next, let's explore an instance of a c-supplemented ring that doesn't fall into category of an SSI-ring.

Example 2.1. Take any prime integer, let's call it 'p'.Now consider the ring $R = \mathbb{Z}_{p^2}$, which happens to be a semilocal ring with a simple radical. According to theorem 2.5, since semisimple modules are susceptible to crumbling we conclude that ${}_R R$ is indeed c-supplemented. However, it's worth nothing that R doesn't qualify as an SSI-ring.. ■

IV. DISCUSSION

Characterizations of c-supplemented rings can be studied and the relationship of these rings with the other ring classes can be investigated.

V. CONCLUSION

In this study, we define c-supplemented modules and explore their manifold properties. In particular, it has been proven that c-supplemented rings are different from SSI-rings.

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