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Solving Problems With Inhomogeneous Boundary Conditions

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Abstract – In this study, heat and wave equations containing ihomogeneous boundary conditions were discussed. For these equations, the data shift method and the Fourier series method were used to reduce the inhomogeneous boundary conditions to homogeneous ones. Methods were compared in order to facilitate the solution and improve the decision phase. These results are used in sectors such as production, health and finance and are effective in increasing productivity.

Keywords – Inhomogeneous, Shifting Method, Boundary Conditions, Heat Equation, Solution

I. Introduction

The wave and heat equation is one of the partial differential equations frequently encountered in the fields of applied mathematics and physics. The wave equation is a second-order partial differential equation of hyperbolic type and models many natural phenomena. For example, events such as gravitational waves, sound waves, light waves and spring motion can be expressed with the wave equation. The heat equation is a piecewise differential equation that describes how heat will distribute over an object over time from a particular location. For these equations, when examining a problem defined in a finite range, the region where the problem is defined is expanded. This expansion extends to both the left and right of the field. Likewise, the necessary conditions for the solution of the equation are determined and a solution is sought using different methods. Finite element method is one of them and there are important studies on this subject [1]-[3].

The Fourier method is also given to solve the problem in a finite range. This method was first developed and generalized by d'Alembert, then by Fourier and Ostrogradski, for the most general case. This method has been used successfully in wave, heat and Laplace equations [4]-[7]. Stability analysis has been studied for systems modeled by [8]. Systems modeled with the wave equation with inverse damping term are given in [9]. With the help of hyperbolic partial differential equation, a backstepping controller that can be applied to linear time-independent delayed systems has been obtained. In [10], the controller is explicit for multi-input linear time-independent delay systems. The control can be generalized to two- and more-dimensional distributed parameter systems. Heat and wave stability analysis and backstep limit value control can be provided for equation systems.

In this study, we will use two different methods, Fourier and separation of variables, to obtain the solutions of these equations and prove the applicability of the results.

II. Materials and Method

In this article, we will see one of the special types of heat equations called inhomogeneous wave equations and the method to find the solution of such equations.

The method of separation of variables can be used to solve non-homogeneous equations. We first consider the heat equation case. Consider the initial boundary problem,

$$u_t - ku_{xx} = f(x,t), \qquad 0 < x < L, \qquad t > 0$$

$$u(0,t) = h(t),\ u(L,t) = p(t),\ t \geq 0,$$

$$u(x,0) = \varphi(x), 0 < x. \quad (1)$$

In this problem, we will use the data shifting method to make the boundaries homogeneous, equivalent to (1).

The idea of this method is to extract a function that satisfies the following. Let's define the function

$$U(x,t) = \left(1 - \frac{x}{l}\right) h(t) + \frac{x}{l} p(t)$$

for which trivially $U(0,t) = h(t)$ and $U(l,t) = p(t)$. But then for the new quantity

$v(x,t) = u(x,t) - U(x,t)$, we have

$$v_t - kv_{xx} = u_t - ku_{xx} - (U_t - kU_{xx})$$
$$= f(x,t) - \left(1 - \frac{x}{l}\right) h'(t) + \frac{x}{l} p'(t)$$

$$v(x,0) = u(x,0) - U(x,0)$$
$$= \varphi(x) - \left(1 - \frac{x}{l}\right) h(0) + \frac{x}{l} p(0)$$

$$v(0,t) = u(0,t) - U(0,t) = h(t) - h(t) = 0,$$

$$v(l,t) = u(l,t) - U(l,t) = p(t) - p(t) = 0.$$

Thus $v(x,t)$ is equivalent to the following boundary value problem with homogeneous boundary conditions.

$$v_t - kv_{xx} = \tilde{f}(x,t), \quad 0 < x < L, \quad t > 0$$

$$v(0,t) = v(L,t) = 0, t \geq 0,$$

$$v(x,0) = \tilde{\varphi}(x), 0 < x, \quad (2)$$

where

$$\tilde{f}(x,t) = f(x,t) - \left(1 - \frac{x}{l}\right) h'(t) + \frac{x}{l} p'(t)$$

$$\tilde{\varphi}(x) = \varphi(x) - \left(1 - \frac{x}{l}\right) h(0) + \frac{x}{l} p(0). \quad (3)$$

Problem (2) is equivalent to (1), so the function $v(x,t)$ can be found, where $\varphi(x)$ with $\tilde{\varphi}(x)$ and $f_n(s)$ is replaced by $\tilde{f_n}(s)$. φen and fen(t) can be found as follows.

After finding the function $v(x,t)$ in series form using (3), the Fourier coefficients of $\varphi(x)$ and $f(x,t)$ are found.

$$v(x,t) = \sum_{n=1}^{\infty} \left[\widetilde{\varphi_n} e^{-k(n\pi/l)^2 t} + \int_0^t e^{-k\left(\frac{n\pi}{l}\right)^2 (s-t)} \tilde{f_n}(s) ds \right] \sin \frac{n\pi x}{l},$$

it gives the solution of problem 2,

$$= \left(1 - \frac{x}{l}\right) h(t) + \frac{x}{l} p(t) + \sum_{n=1}^{\infty} \left[\widetilde{\varphi_n} e^{-k\left(\frac{n\pi}{l}\right)^2 t} + \int_0^t e^{-k\left(\frac{n\pi}{l}\right)^2 (s-t)} \tilde{f_n}(s) ds \right] \sin \frac{n\pi x}{l}.$$

Taking Neumann boundary conditions

$$u_x(0,t) = h(t),\ u_x(l,t) = p(t).$$

Let's consider the following function,

$$A(x,t) = \int U(x,t) dx = (x - \frac{x^2}{2l}) h(t) + \frac{x^2}{2l} p(t)$$

to shift the data, since

$A_x(0,t) = U(0,t) = h(t)$ and

$A_x(l,t) = U(l,t) = p(t)$.

For the inhomogeneous boundary value problem for the wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad 0 < x < l,$$

$$u(0,t) = h(t),\ u(l,t) = p(t),$$

$$u(x,0) = \varphi(x),\ u_t(x,0) = \omega(x), \quad (4)$$

in the problem, data can be shifted in the same way. An alternative method to shifting the data to solve problem 2 is to use the following expansion.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}. \quad (5)$$

Now if we calculate $u_{xx}(x,t)$,

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} \phi_n(t) \sin \frac{n\pi x}{l}. \quad (6)$$

Using Green's second identity here,

$$\phi_n(t) = \frac{2}{l} \int_0^l u_{xx}(x,t) \sin \frac{n\pi x}{l} dx$$
$$= -\frac{2}{l} \int_0^l \left(\frac{n\pi}{l}\right)^2 u(x,t) \sin \frac{n\pi x}{l} dx$$
$$+ \frac{2}{l} \left(u_x \sin \frac{n\pi x}{l} - \frac{n\pi}{l} u \cos \frac{n\pi x}{l}\right).$$

If we use the boundary conditions for u in (2) and consider that the first boundary term disappears

$$\phi_n(t) = -\left(\frac{n\pi}{l}\right)^2 u_n(t) - \frac{2n\pi}{l^2} (-1)^n p(t) + \frac{2n\pi}{l^2} h(t).$$

Using (6) and the last equation, we get,

$\sum_{n=1}^{\infty} \ u'_n(t) - \lambda\left(-\left(\frac{n\pi}{l}\right)^2 u_n(t) - \frac{2n\pi}{l^2}(-1)^n p(t) + \frac{2n\pi}{l^2} h(t)\right) \sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} \ f_n(t) \sin\frac{n\pi x}{l}.$

Using the completeness, we obtain the following ODEs for the coefficients $u_n(t)$

$u'_n(t) + \lambda\left(\frac{n\pi}{l}\right)^2 u_n = f_n(t) - \frac{2n\pi}{l^2}\left((-1)^n p(t) - h(t)\right)$, with $u_n(0)= \varphi_n$

These ODEs can also be solved in the same way using the integration factor.

$u_n(t) = \varphi_n e^{-k(n\pi/l)^2 t} + \int_0^t e^{k\left(\frac{n\pi}{l}\right)^2 (s-t)} \left(f_n(s) - \frac{2n\pi}{l^2}(-1)^n p(s) - h(s)\right) ds.$

has a solution function. For the inhomogeneous wave problem (4), this method will give the following ODEs for the coefficients.

$u''_n + c^2\left(\frac{n\pi}{l}\right)^2 u_n = \left(f_n(t) - \frac{2n\pi}{l^2}(-1)^n p(t) - h(t)\right),$

with the initial conditions

$u_n(0)= \varphi_n, u'_n(0)= \omega_n.$ (7)

Solving these ODEs by variation of parameters, one can find the solution to (4) as the series (5). Now, let's use the Fourier series, with which we successfully solved boundary value problems, in our search for solutions to non-homogeneous problems.

Consider the following boundary value problem for the Dirichlet inhomogeneous heat equation,

$$u_t - ku_{xx} = f(x,t), \qquad 0 < x < l, \qquad t > 0$$
$$u(0,t) = u(l,t) = 0$$
$$u(x,0) = \varphi(x),$$

Let's approach the problem in series form as a solution in (5). In this Fourier sine series, the coefficients will change with the t variable. Such an expansion always exists due to the completeness of the set of eigenfunctions $\{\sin\frac{n\pi x}{l}\}$. The coefficients will then be given by the Fourier sine coefficients formula

$u_n(t) = \frac{2}{l}\int_0^l u(x,t)\sin\frac{n\pi x}{l}\,dx.$

We can similarly expand the source function,

$f(x,t) = \sum_{n=1}^{\infty} \ f_n(t) \sin\frac{n\pi x}{l},$ where

$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx.$ (8)

Now, since we are looking for a twice differentiable function $u(x,t)$ that satisfies the homogeneous

Dirichlet boundary conditions, we can differentiate the Fourier series (5) term by term to obtain

$u_{xx}(x,t) = -\sum_{n=1}^{\infty} \ u_n(t)\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}.$ (9)

To be able to differentiate twice, we need to guarantee that u_x satisfies homogeneous Neumann conditions; This is achieved by taking the single extension of u into the range $(-l, 0)$.

We can also differentiate series (5) with respect to t,

$u_t(x,t) = \sum_{n=1}^{\infty} \ u'_n(t) \sin\frac{n\pi x}{l}.$ (10)

$u_t(x,t)$ are Fourier coefficients,

$\frac{2}{l}\int_0^l u_t(x,t) \sin\frac{n\pi x}{l}\,dx = \frac{\partial}{\partial t}\frac{2}{l}\int_0^l u(x,t) \sin\frac{n\pi x}{l}\,dx = u'_n(t)$

If we differentiate under the integral and substitute (10) and (9) into use Equation (8) and also taking into account completeness, we get,

$$u_n(t) = u(0)e^{-k\left(\frac{n\pi}{l}\right)^2 t} + e^{-k\left(\frac{n\pi}{l}\right)^2 t}\int_0^l e^{-k\left(\frac{n\pi}{l}\right)^2 s} f_n(s)ds \qquad (11)$$

If we use the initial condition,

$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} \ u_n(0) \sin\frac{n\pi x}{l},$

$+ \ u_n(0)= \omega_n= \frac{2}{l}\int_0^l \varphi(x) \sin\frac{n\pi x}{l}\,dx.$

Then the solution can be written in series form as,

$$u(x,t) = \sum_{n=1}^{\infty}\left[\varphi_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} + \int_0^t e^{-k\left(\frac{n\pi}{l}\right)^2 (s-t)} f_n(s)ds\right]\sin\frac{n\pi x}{l}.$$

where φ_n and f_n are the Fourier coefficients of $f(x,t)$ and can be found from (8) and (11), respectively. The first coefficient term in the above series comes from the homogeneous heat equation, while the second term It is obtained by Neumann boundary conditions for the inhomogeneous heat equation, The only difference is that a cosine series solution is sought instead of a sine series (5). Boundary value problem for inhomogeneous wave equation of (4),

$$u_n'' + c^2\left(\frac{n\pi}{l}\right)^2 u_n = f_n(t)$$

with the initial conditions in (7). mThese ODEs can be solved explicitly using variation of parameters to obtain the coefficients $u_n(t)$.

III. RESULTS

We sought solutions to inhomogeneous heat and wave equations by using eigenfunctions corresponding to homogeneous boundary value problems. This leads to ODEs for the coefficients of a series in terms of eigenfunctions.

IV. DISCUSSION

We studied two methods for solving problems with inhomogeneous boundary conditions. The first of these is to shift the boundary data to reduce the problem to a homogeneous situation. The second method is to seek a solution directly as a series in terms of the eigenfunctions of the relevant homogeneous problem. Since boundary terms emerge when differentiating Fourier series, ODEs can only be reached after dealing with these boundary terms.

V. CONCLUSION

As a result of the Fourier method, the solution is the infinite sum and it is obtained in series. In the non-homogeneous case, the considered partial differential equation turns into a Sturm-Liouville problem. The Sturm-Liouville problem is open to extensive investigation. For this reason, the Fourier method is generally preferred in solving inhomogeneous problems in the finite range.

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