A Study on Lacunary Summability of Order α with respect to Modulus Function for Fuzzy Variables in Credibility Spaces

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Abstract – The main aim of this study is to investigate strongly lacunary summable and lacunary statistically convergent fuzzy variable sequences (briefly FVS) by utilizing modulus functions f and s under some conditions and orders $\gamma, \rho \in (0,1]$ such that $\gamma \leq \rho$. In addition, we obtain some inclusion relations between these concepts.

Keywords – Lacunary Sequence, Lacunary Summability, Modulus Function, Fuzzy Variable Sequence, Credibility Space

I. Introduction

Fuzzy theory was pioneered by Zadeh [1] in 1965. A fuzzy variable (FV) is a function that maps from a credibility space to a set of real values. The convergence of FVs is a key component of credibility theory, which may be applied to real-world engineering and financial challenges. Kaufmann [2] has conducted research on FVs, possibility distributions, and membership functions. Several specific contents have been explored since Liu began his investigation of credibility theory (see [3-9]). Given the relevance of sequence convergence in credibility theory, Liu [5] proposed four forms of convergence concepts for FVs: credibility convergence, almost certainly convergence, mean convergence, and distribution convergence.

Fast [10] presented statistical convergence for real sequences as an extension of ordinary convergence. Gadjiev and Orhan [11] put forward the order of statistical convergence of a sequence of operators and then Çolak [12] worked the order of statistical convergence for a sequence of numbers. Lacunary statistical convergence was studied by Fridy and Orhan [13]. Significant studies on this topic can be examined (see [14-15]). Nakano [16] investigated the idea of a modulus function. By utilizing a modulus function, several authors constructed new sequence spaces (see [17-20]).

A set function Cr is credibility measure if it provides the subsequent axioms: Let H be a nonempty set, and Θ be a nonempty set, and $\mathcal{P}(\Theta)$ be the power set of Θ (i.e., the largest algebra over Θ). All element in $\mathcal{P}$ is named an event. For any $A \in \mathcal{P}(\Theta)$, Liu and Liu [6] presented a credibility measure $\mathrm{Cr}(A)$ to indicate the chance that fuzzy event A occurs. Li and Liu [3] proved that a set function $\mathrm{Cr}(.)$ a credibility measure iff

Axiom i. $\mathrm{Cr}(\Theta) = 1$;

Axiom ii. $\mathrm{Cr}(A) \leq \mathrm{Cr}(B)$ whenever $A \subset B$;

Axiom iii. Cr is self-dual, i.e., $\mathrm{Cr}(A) + \mathrm{Cr}(A^c) = 1$, for any $A \in \mathcal{P}(\Theta)$;

Axiom iv. $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is called a credibility space. A fuzzy variable is put forward by Liu and Liu [3] as function from the credibility space to the set of real numbers.

Now, we serve the concepts of investigate strongly lacunary summable and lacunary statistically convergent FVS by utilizing modulus functions f and s under some conditions and orders $\gamma, \rho \in (0,1]$ such that $\gamma \leq \rho$, and obtain some features of these concepts.

II. MAIN RESULTS

In this section, we present the relations between $N_\theta^\gamma(s)$ and $N_\theta^\rho(f)$, $N_\theta^\rho(s)$ and $N_\theta^\gamma(f)$, $S_\theta^\rho(s)$ and $N_\theta^\gamma(f), N_\theta^\rho(g)$ and $\ell_\infty \cap S_\theta^\gamma(f)$ for FVS in credibility spaces, where f and s are modulus functions under some conditions and $\gamma, \rho \in (0,1]$ such that $\gamma \leq \rho$. Throughout the article, let f, s be modulus functions, $\theta = (k_r)$ be a lacunary sequence, $\mu, \mu_1, \mu_2, \ldots$ be fuzzy variables identified on credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$, and take $\gamma, \rho \in (0,1]$.

Definition 2.1. A FVS $\{\mu_k\}$ is named to be strongly $N_\theta^\gamma(f)$-summable (or strongly f-lacunary summable) of order γ to the FV μ provided, there exists a $A \in \mathcal{P}(\Theta)$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^\gamma} \sum_{k\in I_r} f(|\mu_k(\theta) - \mu(\theta)|) = 0$$

for all $\theta \in$ A. In this case, we denote $\mu_k \to \mu\left(N_\theta^\gamma(f)\right)$ or $N_\theta^\gamma(f) - \lim \mu_k = \mu$. The sets of strongly $N_\theta^\gamma(f)$-summable FVS can be demonstrated by $N_\theta^\gamma(f)$. Namely,

$$N_\theta^\gamma(f) = \left\{\{\mu_k\}: \lim_{r\to\infty} \frac{1}{h_r^\gamma} \sum_{k\in I_r} f(|\mu_k(\theta) - \mu(\theta)|) = 0 \text{ for some FV } \mu\right\}.$$

In this definition, we emphasize that the modulus function f need not to be unbounded.

Theorem 2.1. Assume f and s be modulus functions, $\gamma, \rho \in (0,1]$ so that $\gamma \leq \rho$. When

$$\sup_{w\in(0,\infty)} \frac{f(w)}{s(w)} < \infty$$

then $N_\theta^\gamma(s) \subset N_\theta^\rho(f)$.

Proof. Take $t = \sup_{w\in(0,\infty)} \frac{f(w)}{s(w)} < \infty$. At that time, we get $0 < \frac{f(w)}{s(w)} \leq t$ and hence $f(w) \leq ts(w)$ for any $w \geq 0$. It is obvious that $t > 0$ and if $N_\theta^\gamma(s) - \lim \mu_k = \mu$, then

$$\frac{1}{h_r^\gamma} \sum_{k\in I_r} f(|\mu_k(\theta) - \mu(\theta)|) \leq \frac{1}{h_r^\gamma} \sum_{k\in I_r} ts(|\mu_k(\theta) - \mu(\theta)|)$$

for all $\theta \in$ A, where $A \in \mathcal{P}(\Theta)$. Since $\gamma \leq \rho$, we obtain

$$\frac{1}{h_r^\rho} \sum_{k\in I_r} f(|\mu_k(\theta) - \mu(\theta)|) \leq t \frac{1}{h_r^\gamma} \sum_{k\in I_r} s(|\mu_k(\theta) - \mu(\theta)|)$$

for all $\theta \in$ A. Getting the limits on both sides as $r \to \infty$, we acquire that $\{\mu_k\} \in N_\theta^\gamma(s)$ gives $\{\mu_k\} \in N_\theta^\rho(f)$.

Remark 2.1. The following example demonstrates that the inclusion $N_\theta^\gamma(s) \subset N_\theta^\rho(f)$ is strict.

Example 2.1. Choose $\gamma = \rho = 1$ and identify FVS $\{\mu_k\}$ as μ_k to be $\left[\sqrt{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\mu_k = 0$ if not. When we establish the modulus functions $f(w) = \frac{w}{w+1}$ and $s(w) = w$, then $\sup_{w\in(0,\infty)} \frac{f(w)}{s(w)} = 1 < \infty$ and so $N_\theta^\gamma(s) \subset N_\theta^\rho(f)$ by *Theorem 2.1*. With the aid of the $f(0) = 0$ equality, we get

$$\frac{1}{h_r^\rho} \sum_{k\in I_r} f(|\mu_k(\theta)|) = \frac{1}{h_r}\left[\sqrt{h_r}\right] f\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r\left(\left[\sqrt{h_r}\right] + 1\right)}$$

for all $\theta \in$ A. Getting the limits as $r \to \infty$, we obtain that $N_\theta^\rho(f) - \lim \mu_k = 0$. Hence, $\{\mu_k\} \in N_\theta^\rho(f)$. However, since

$$\frac{1}{h_r^\gamma} \sum_{k \in I_r} s(|\mu_k(\theta)|) = \frac{1}{h_r} \left[\sqrt{h_r}\right] s\left(\left[\sqrt{h_r}\right]\right) = \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]}{h_r}$$

and $\frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r} \to 1$ as $r \to \infty$, we have $\{\mu_k\} \notin N_\theta^\gamma(s)$. As a result $\{\mu_k\} \in N_\theta^\rho(f) - N_\theta^\gamma(s)$ and the inclusion $N_\theta^\gamma(s) \subset N_\theta^\rho(f)$ is strict.

Corollary 2.1. Assume f and s be modulus functions, $\gamma, \rho \in (0,1]$ so that $\gamma \leq \rho$.

1. When $\sup_{w \in (0,\infty)} \frac{f(w)}{s(w)} < \infty$, then $N_\theta^\gamma(s) \subset N_\theta^\gamma(f)$.
2. When $\sup_{w \in (0,\infty)} \frac{f(w)}{s(w)} < \infty$, then $N_\theta(s) \subset N_\theta(f)$.
3. $N_\theta^\gamma(f) \subset N_\theta^\rho(f)$.
4. $N_\theta^\gamma \subset N_\theta^\rho$.

Theorem 2.2. If

$$\inf_{w \in (0,\infty)} \frac{f(w)}{s(w)} > 0,$$

then $N_\theta^\gamma(f) \subset N_\theta^\rho(s)$ and the inclusion is strict.

Proof. Take $t = \inf_{w \in (0,\infty)} \frac{f(w)}{s(w)} > 0$. So that $\frac{f(w)}{s(w)} \geq t$ and $ts(w) \leq f(w)$ for all $w \geq 0$. If $N_\theta^\gamma(f) - \lim \mu_k = \mu$, then

$$\frac{1}{h_r^\gamma} \sum_{k \in I_r} s(|\mu_k(\theta) - \mu(\theta)|) \leq \frac{1}{h_r^\gamma} \sum_{k \in I_r} \frac{1}{t} f(|\mu_k(\theta) - \mu(\theta)|)$$

Since $\gamma \leq \rho$, we get

$$\frac{1}{h_r^\rho} \sum_{k \in I_r} s(|\mu_k(\theta) - \mu(\theta)|) \leq \frac{1}{h_r^\gamma} \sum_{k \in I_r} \frac{1}{t} f(|\mu_k(\theta) - \mu(\theta)|).$$

Getting the limits on both sides as $r \to \infty$, we obtain $N_\theta^\rho(s) - \lim \mu_k = \mu$ and so $\{\mu_k\} \in N_\theta^\rho(s)$. For the strict inclusion, the FVS of Example 2.1. with functions $s(w) = \frac{w}{w+1}$ and $f(w) = w$ serve the purpose in the case $\gamma = \rho = 1$.

Corollary 2.2. Assume f and s are modulus functions, $\gamma, \rho \in (0,1]$ so that $\gamma \leq \rho$.

1. When $\inf_{w \in (0,\infty)} \frac{f(w)}{s(w)} > 0$, then $N_\theta^\gamma(f) \subset N_\theta^\gamma(s)$.
2. When $\inf_{w \in (0,\infty)} \frac{f(w)}{s(w)} > 0$, then $N_\theta(f) \subset N_\theta(s)$.
3. $N_\theta^\gamma(f) \subset N_\theta^\rho(f)$.
4. $N_\theta^\gamma \subset N_\theta^\rho$.

Corollary 2.3. If

$$0 < \inf_{w \in (0,\infty)} \frac{f(w)}{s(w)} \leq \sup_{w \in (0,\infty)} \frac{f(w)}{s(w)} < \infty$$

then $N_\theta^\gamma(f) = N_\theta^\gamma(s)$.

Corollary 2.4. If $\sup_{w \in (0,\infty)} \frac{f(w)}{w} < \infty$, then $N_\theta^\gamma \subset N_\theta^\rho(s)$ for any $\gamma, \rho \in (0,1]$ so that $\gamma \leq \rho$.

Corollary 2.5. If $\sup_{w \in (0,\infty)} \frac{f(w)}{w} < \infty$, then $N_\theta^\gamma \subset N_\theta^\gamma(f)$ for any $\gamma \in (0,1]$.

Corollary 2.6. If $\inf_{w \in (0,\infty)} \frac{f(w)}{w} > 0$, then $N_\theta^\gamma(f) \subset N_\theta^\rho$ for any $\gamma, \rho \in (0,1]$ such that $\gamma \leq \rho$.

Corollary 2.7. If $\inf_{w \in (0,\infty)} \frac{f(w)}{w} > 0$, then $N_\theta^\gamma(f) \subset N_\theta^\gamma$ for any $\gamma \in (0,1]$.

Corollary 2.8. If

$$0 < \inf_{w \in (0,\infty)} \frac{f(w)}{w} \leq \sup_{w \in (0,\infty)} \frac{f(w)}{w} < \infty,$$

then $N_\theta^\gamma(f) = N_\theta^\gamma$ for any $\gamma \in (0,1]$.

Theorem 2.3. When $\inf_{w \in (0,\infty)} \frac{f(w)}{s(w)} > 0$ and $\lim_{w \to \infty} \frac{s(w)}{w} > 0$, then all strongly $N_\theta^\gamma(f)$-summable FVS is $S_\theta^\rho(s)$-convergent.

Proof. Presume that $t = \inf_{w \in (0,\infty)} \frac{f(w)}{s(w)} > 0$. Then $\frac{f(w)}{s(w)} \geq t$ and hence $ts(w) \leq f(w)$ for all $w \geq 0$. If

$N_\theta^\gamma(f) - \lim\mu_k = \mu$ and $\gamma, \rho \in (0,1]$ so that $\gamma \leq \rho$, then

$$\frac{1}{h_r^\gamma}\sum_{k\in I_r} f(|\mu_k(\theta)-\mu(\theta)|)$$
$$\geq t\frac{1}{h_r^\gamma}\sum_{k\in I_r} s(|\mu_k(\theta)-\mu(\theta)|)$$
$$\geq t\frac{1}{h_r^\rho}\sum_{k\in I_r} s(|\mu_k(\theta)-\mu(\theta)|)$$
$$= t\frac{1}{h_r^\rho}\sum_{\substack{k\in I_r\\ |\mu_k(\theta)-\mu(\theta)|\geq\varepsilon}} s(|\mu_k(\theta) - \mu(\theta)|) + t\frac{1}{h_r^\rho}\sum_{\substack{k\in I_r\\ |\mu_k(\theta)-\mu(\theta)|<\varepsilon}} s(|\mu_k(\theta) - \mu(\theta)|)$$
$$\geq t\frac{1}{h_r^\rho}\sum_{\substack{k\in I_r\\ |\mu_k(\theta)-\mu(\theta)|\geq\varepsilon}} s(|\mu_k(\theta) - \mu(\theta)|)$$
$$\geq t\frac{1}{h_r^\rho}|\{k\in I_r: |\mu_k(\theta)-\mu(\theta)| \geq \varepsilon\}|s(\varepsilon).$$

for all $\theta \in$ A. As $|\{k\in I_r: |\mu_k(\theta)-\mu(\theta)|\geq\varepsilon\}|$ is a positive integer, we obtain

$$\frac{1}{h_r^\gamma}\sum_{k\in I_r} f(|\mu_k(\theta)-\mu(\theta)|)$$
$$\geq \frac{1}{h_r^\rho}s(|\{k\in I_r: |\mu_k(\theta)-\mu(\theta)|\geq\varepsilon\}|)\frac{s(\varepsilon)}{s(1)}t =$$
$$= \frac{s(|\{k\in I_r: |\mu_k(\theta)-\mu(\theta)|\geq\varepsilon\}|)}{s(h_r^\rho)}\frac{s(h_r^\rho)}{h_r^\rho}\frac{s(\varepsilon)}{s(1)}t.$$

Getting the limits on both sides as $r\to\infty$, we obtain that $\{\mu_k\}\in N_\theta^\gamma(f)$ means $\{\mu_k\}\in S_\theta^\rho(s)$ since $\lim_{w\to\infty}\frac{s(w)}{w} > 0$.

Remark 3.2. Generally, contrary of the Theorem 2.3 could be impossible. Following example demonstrates this situation.

Example 2.2. Establish the FVS $\{\mu_k\}$ as in Example 2.1 and also take $s(w) = f(w) = w$. Hence $\inf_{w\in(0,\infty)}\frac{f(w)}{s(w)} > 0$ and $\lim_{w\to\infty}\frac{s(w)}{w} > 0$. If we assume $0 < \gamma \leq \frac{1}{2} < \rho \leq 1$, then for any $\varepsilon > 0$, we get

$$\lim_{r\to\infty}\frac{1}{s(h_r^\rho)}s(|\{k\in I_r: |\mu_k(\theta)|\geq\varepsilon\}|)$$
$$= \lim_{r\to\infty}\frac{[\sqrt{h_r}]}{h_r^\rho} = 0.$$

Therefore, $\{\mu_k\}\in S_\theta^\rho(s)$. However, since

$$\lim_{r\to\infty}\frac{1}{h_r^\gamma}\sum_{k\in I_r} f(|\mu_k(\theta)|) = \lim_{r\to\infty}\frac{[\sqrt{h_r}][\sqrt{h_r}]}{h_r^\gamma} = \infty,$$

as a result $\{\mu_k\}\notin N_\theta^\gamma(f)$.

Corollary 2.9. Assume f is an unbounded modulus, $\gamma,\rho\in(0,1]$ so that $\gamma\leq\rho$. If $\lim_{w\to\infty}\frac{f(w)}{w} > 0$, then all strongly $N_\theta^\gamma(f)$-convergent FVS is $S_\theta^\rho(f)$-convergent.

Corollary 2.10. Assume f and g are unbounded modulus functions, $\gamma\in(0,1]$. If $\inf_{w\in(0,\infty)}\frac{f(w)}{s(w)} > 0$ and $\lim_{w\to\infty}\frac{s(w)}{w} > 0$, then all strongly $N_\theta^\gamma(f)$-convergent FVS is $S_\theta^\gamma(s)$-convergent.

Corollary 2.11. If $\inf_{u\in(0,\infty)}\frac{f(u)}{u} > 0$, then all strongly $N_\theta^\gamma(f)$ convergent FVS is S_θ^γ-convergent and also S_θ-convergent.

Theorem 2.4. Let f and g be any unbounded modulus functions, $0 < \alpha \leq \beta \leq 1$, and assume $\theta = (k_r)$ and $\vartheta = (t_r)$ are lacunary sequences so that $I_r \subset I_r{}'$ for all $r\in\mathbb{N}$. If $\lim_{r\to\infty}\frac{v_r}{h_r^\rho} = 1$ and $\sup_{w\in(0,\infty)}\frac{s(w)}{w} < \infty$, then all bounded and $S_\theta^\gamma(f)$-convergent FVS is strongly $N_\vartheta^\rho(s)$-convergent, namely,

$$\ell_\infty \cap S_\theta^\gamma(f) \subset N_\vartheta^\rho(s).$$

where $I_r = (k_{r-1}, k_r], I_r{}' = (t_{r-1}, t_r], h_r = k_r - k_{r-1}, v_r = t_r - t_{r-1}$.

Proof. Take $0 < \alpha \leq \beta \leq 1$. Let $\{\mu_k\} \in \ell_\infty \cap S_\theta^\gamma(f)$ and $S_\theta^\gamma(f) - \lim\mu_k = \mu$. To confirm that

$\{\mu_k\} \in N_{\vartheta}^{\rho}(s)$, we have to demonstrate that $S_{\theta}^{\gamma}(f) \subset S_{\theta}^{\gamma}$. Considering f is a modulus and $S_{\theta}^{\gamma}(f) - \lim \mu_k = \mu$, for all $q \in \mathbb{N}$ there is a $r_0 \in \mathbb{N}$ so that, if $r > r_0$, we obtain

$$f(|\{k \in I_r: |\mu_k(\theta) - \mu(\theta)| \geq \varepsilon\}|) \leq \frac{1}{q} f(h_r^{\gamma}) \leq \frac{1}{q} q f\left(\frac{h_r^{\gamma}}{q}\right) = f\left(\frac{h_r^{\gamma}}{q}\right)$$

for any $\varepsilon > 0$. Hence,

$$\frac{1}{h_r^{\gamma}} |k \in I_r: |\mu_k(\theta) - \mu(\theta)| \geq \varepsilon| \leq \frac{1}{q}.$$

It follows that $S_{\theta}^{\gamma}(f) \subset S_{\theta}^{\gamma}$ and so $\ell_{\infty} \cap S_{\theta}^{\gamma}(f) \subset \ell_{\infty} \cap S_{\theta}^{\gamma}$. Since $\lim_{r\to\infty} \frac{v_r}{h_r^{\rho}} = 1$, we get $\ell_{\infty} \cap S_{\theta}^{\gamma} \subset N_{\vartheta}^{\rho}$. Thereby $N_{\vartheta}^{\rho} \subset N_{\vartheta}^{\rho}(s)$ since $\sup_{w\in(0,\infty)} \frac{s(w)}{w} < \infty$. As a result, $\ell_{\infty} \cap S_{\theta}^{\gamma}(f) \subset N_{\vartheta}^{\rho}(s)$.

Remark 2.3. The inclusion $\ell_{\infty} \cap S_{\theta}^{\gamma}(f) \subset N_{\vartheta}^{\rho}(s)$ is strict.

Example 2.3. Let the lacunary sequence $\theta = (k_r)$ be provided and $\vartheta = \theta$. Identify the FVS (μ_k) as μ_k to be $\left[\sqrt[3]{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\mu_k = 0$ if not. In addition, establish the modulus functions $f(w) = s(\mathrm{w}) = w$. If we take $0 < \gamma \leq \frac{1}{2}$ and $\rho = 1$, then $\lim_{r\to\infty} \frac{v_r}{h_r^{\rho}} = 1$ and $\sup_{w\in(0,\infty)} \frac{s(w)}{w} = 1 < \infty$. Since $\vartheta = \theta$, then for any $r \in \mathbb{N}$, we obtain

$$\frac{1}{v_r^{\rho}} \sum_{k\in I_{r'}} s(|\mu_k(\theta)|) = \frac{1}{v_r^{\rho}} \sum_{k\in I_{r'}} s\left(\left[\sqrt[3]{v_r}\right]\right) = \frac{\left[\sqrt{v_r}\right]\left[\sqrt[3]{v_r}\right]}{v_r}.$$

Since $\frac{[\sqrt{v_r}][\sqrt[3]{v_r}]}{v_r} \to 0$ as $r \to \infty$, then $(\mu_k) \in N_{\vartheta}^{\rho}(s)$. However, for all $\varepsilon > 0$, we can write

$$\frac{1}{f(h_r^{\gamma})} f(|\{k \in I_r: |\mu_k(\theta)| \geq \varepsilon\}|) = \frac{f\left(\left[\sqrt{h_r}\right]\right)}{f(h_r^{\gamma})} = \frac{\left[\sqrt{h_r}\right]}{h_r^{\gamma}}$$

So, $(\zeta_k) \notin S_{\theta}^{\gamma}(f)$ since $\frac{[\sqrt{h_r}]}{h_r^{\gamma}} \to \infty$ as $r \to \infty$ for $0 < \gamma < \frac{1}{2}$ and $\frac{[\sqrt{h_r}]}{h_r^{\gamma}} \to 1$ as $r \to \infty$ for $\gamma = \frac{1}{2}$. As a result, the inclusion $\ell_{\infty} \cap S_{\theta}^{\gamma}(f) \subset N_{\vartheta}^{\rho}(s)$ is strict.

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