All Sciences Proceedings
http://as-proceeding.com/

1st International Conference on Modern and Advanced Research
July 29-31, 2023 : Konya, Turkey

ICMAR
https://as-proceeding.com/index.php/icmar

SOME NEW GENERALIZED CLASSES OF SEQUENCES OF FUZZY NUMBERS DEFINED BY A ORLICZ FUNCTION II

Ayhan Esi[*]

[1]*Dept.of Basic Eng.Sci., Engineering Faculty, Malatya Turgut Ozal University, Türkiye*

[*]*(ayhan.esi@ozal.edu.tr, aesi23@hotmail..com) Emails of the corresponding author*

Abstract – In this paper we introduce the concept of strongly $A_\sigma^{\lambda(p)}$ – convergence of fuzzy numbers with respect to an Orlicz function and examine some properties of the resulting sequence spaces and $\lambda(\sigma)$-statistical convergence. It also shown that if a sequence of fuzzy numbers is strong $\lambda(\sigma)$ convergent with respect to an Orlicz function then it is $\lambda(\sigma)$-statistically convergent.

Keywords – Orlicz Function,Paranorm, De La Vallee-Poussin Means, Fuzzy Number.

I. INTRODUCTION

The concept of paranorm is closely related to linear metric spaces.It is a generalization of that of absolute value. Let X be a linear space. A function $g : X \to R$ is called paranorm, if

(i) $g(0)=0$,(ii) $g(x)\geq 0$, for all $x \in X$, (iii) $g(-x)=g(x)$, for all $x \in X$, (iv) $g(x+y)\leq g(x)+g(y)$, for all $x, y \in X$, and (v) if (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ $(n \to \infty)$ and (x_n) a sequence of vectors with $g(x_n - x)\to 0$ $(n \to \infty)$, then $g(\alpha_n x_n - \alpha x)\to 0$ $(n \to \infty)$.This property is called continuity of multiplication by scalars.The space X is called the paranormed space with the paranorm g.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1,2,3, \ldots$. A continuous linear functional φ on l_∞ , the set of all bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$, where e=(1,1,1,…) and

(iii) $\varphi(\{x_{\sigma(n)}\}) = \varphi(\{x_n\})$ $all\ x = (x_n) \in l_\infty$.

For certain kinds of mappings σ, every invariant mean φ extends the limit functional on the space c,the set of all convergent sequences, in the sense that $\varphi(x) = lim x$ for all $x = (x_n) \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ-means are equal.

An Orlicz function is a function M: $[0, \infty[\to [0, \infty[$, which is continuous, non-decreasing and convex with M(0)=0, M(x) > 0 for x > 0 and M(x)$\to \infty$ $as\ x \to \infty$.

An Orlicz function is said to satisfy Δ_2-condition for all values of u, if there exists a constant K > 0, such that M(2u) ≤ KM(u), u ≥ 0.

Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x = (x_k): \quad \sum_k M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad for\ some\ \rho > 0 \right\}.$$

The space l_M with the norm

$$\|x\| = Inf\left\{ \rho > 0: \quad \sum_k M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p, 1 \le p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [9], Esi,Isik and Esi [10] , Esi [11], Esi and Et [12], Esi[13] and many others.

The purpose of this paper is to introduce and study the concepts of strong $A_\sigma^{\lambda(p)}$ convergence of fuzzy numbers with respect to an Orlicz function and $\lambda(\sigma)$-istatistical convergence and some relations between them.

Let $p=(p_k)\in \ell_\infty$, then the following well-known inequality will be used the paper:
For sequences (a_k) and (b_k) of complex numbers we have

$$| a_k + b_k |^{p_k} \le K(| a_k |^{p_k} + | b_k |^{p_k})$$

where $K = max\,(1,2^{H-1})$ and $H = \sup_k p_k$.

We now give here a brief introduction about the sequences of fuzzy numbers (see [1] and [6])

Let D denote the set of all bounded intervals $A = [\underline{A}, \overline{A}]$ on the real line R. For $A, B \in D$, define

$$A \le B \quad if\ and\ only\ if\ \underline{A} \le \underline{B} \quad and \quad \overline{A} \le \overline{B},$$

$$d(A,B) = \max\left\{ |\underline{A} - \underline{B}|,\ |\overline{A} - \overline{B}| \right\}.$$

Then it can be easily seen that d defines a metric on D and (D,d) is a complete metric space [1]. Also the relation $\le$ is a partial order on D.

A fuzzy number is a fuzzy subset of the real line R which is bounded, convex and normal.Let L(R) have compact support, i.e. if $X \in L$(R) then for any $\alpha \in [0,1]$, X^α is compact,where

$$X^\alpha = \{ t : X(t) \ge \alpha \ \ if \ \ \alpha \in (0,1] \}$$

and

$$X^0 = cl\left(\{ t \in R : X(t) > \alpha \ \ if \ \ \alpha = 0 \}\right),$$

where $cl(A)$ is the closure of A.

The set R of real numbers can be embedded in L(R) if we define $\bar{r} \in L$(R) by

$$\bar{r}(t) = \begin{matrix} 1, & if\ t = r \\ 0, & if\ t \ne r \end{matrix} .$$

The additive identity and multiplicative identity of L(R) are denoted by $\bar{0}$ and $\bar{1}$, respectively. Then the arithmetic operations on L(R) are defined as follows:

$$(X \oplus Y)(t) = \sup\{ X(s) \wedge Y(t-s) \}, t \in R;$$

$$(X \Theta Y)(t) = \sup\{ X(s) \wedge Y(s-t) \}, t \in R;$$

$$(X \otimes Y)(t) = \sup\{ X(s) \wedge Y(t/s) \}, t \in R;$$

$$(X / Y)(t) = \sup\{ X(st) \wedge Y(s) \}, t \in R.$$

These operations can be defined in terms of α-level sets as follows:

$$[X \oplus Y]^\alpha = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha],$$

$$[X \Theta Y]^\alpha = [a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha];$$

$$[X \otimes Y]^{\alpha} = \left[\min_{i\in\{1,2\}} a_i^{\alpha} b_i^{\alpha}, \max_{i\in\{1,2\}} a_i^{\alpha} b_i^{\alpha}\right],$$

$$\left[X^{-1}\right]^{\alpha} = \left[\left(a_2^{\alpha}\right)^{-1}, \left(a_1^{\alpha}\right)^{-1}\right], \quad a_i^{\alpha} > 0$$

for each $0 < \alpha \le 1$.

For r in R and X in L(R), the product rX is defined as follows:

$$rX(t) = \begin{cases} X\left(r^{-1}t\right) & if \ r \ne 0 \\ 0 & if \quad r = 0 \end{cases}.$$

Define a map $\bar{d} : L(R) x L(R) \to R_{+} \cup \{0\}$ by $\bar{d}(X,Y) = \sup_{0\le\alpha\le1} d\left(X^{\alpha}, Y^{\alpha}\right)$. For $X, Y \in L(\mathrm{R})$ define $X \le Y$ if and only if $X^{\alpha} \le Y^{\alpha}$ for any $\alpha \in [0,1]$. It is known that (L(R), $\bar{d}$) is a complete metric space [7].

A metric on L(R) is said to be a translation invariant if $\bar{d}(X+Z, Y+Z) = \bar{d}(X,Y)$ for $X, Y, Z \in L(\mathrm{R})$.

LEMMA. [2]. If $\bar{d}$ is a translation invariant metric on L(R), then

(i) $\bar{d}\left(X+Y, \bar{0}\right) \le \bar{d}\left(X, \bar{0}\right) + \bar{d}\left(Y, \bar{0}\right)$,

(ii) $\bar{d}\left(\lambda X, \bar{0}\right) \le |\lambda| \bar{d}\left(X, \bar{0}\right), |\lambda| > 1$.

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded.

A sequence $X = (X_k)$ of fuzzy numbers is said to be converge to a fuzzy number X_o if for every $\varepsilon > 0$ there is a positive integer n_0 such that $\bar{d}(X_k, X_o) < \varepsilon$ for $k > n_0$.

II. RESULTS

Let $\Lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$, $I_n = [n - \lambda_n + 1, n]$. Let M be an Orlicz function, $p = (p_k)$ be any sequence of strictly positive real numbers and $X = (X_k)$ be sequence of fuzzy numbers, then for some $\rho > 0$;

Esi [14] has defined the following classes of sequences of fuzzy numbers:

$$F_o[A_\sigma, M, \lambda, p] = \Bigg\{ X$$

$$= (X_k): \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} a_{mk} \left[M \left(\frac{\bar{d}\left(X_{\sigma^k(m)}, \bar{0}\right)}{\rho} \right.\right.$$

$$F[A_\sigma, M, \lambda, p] = \Bigg\{ X = (X_k): \lim_{n\to\infty} \frac{1}{\lambda_n} \Sigma_{k\in I_n} a_{mk} \left[M \left(\frac{\bar{d}(X}{} \right.\right.$$

$$0, uniformly \ in \ m \Bigg\}$$

,

$$F_\infty[A_\sigma, M, \lambda, p] = \Bigg\{ X = (X_k): \sup_{n,m} \frac{1}{\lambda_n} \Sigma_{k\in I_n} a_{mk} \left[M \left(\frac{\bar{d}(}{} \right.\right.$$

.

We denote $F[A_\sigma, M, \lambda, p]$, $F_o[A_\sigma, M, \lambda, p]$ and $F_\infty[A_\sigma, M, \lambda, p]$ as $F[A_\sigma, M, \lambda]$, $F_o[A_\sigma, M, \lambda]$ and $F_\infty[A_\sigma, M, \lambda]$, when $p_k = 1$ for all k. If $X = (X_k) \in F[A_\sigma, M, \lambda, p]$ we say that $X = (X_k)$ is of strongly $A_\sigma^{\lambda(p)}$- convergent to fuzzy number X_0 with respect to the Orlicz function M. If ,M(x)=x, A=(C,1) matrix order 1, $\lambda_n = n$ for all n, and $\sigma(n) = n$, then $F[A_\sigma, M, \lambda, p]$= $F[p]$, $F_o[A_\sigma, M, \lambda, p]$=$F_o[p]$ and $F_\infty[A_\sigma, M, \lambda, p]$=$F_\infty[p]$, which were defined by Mursaleen and Basarir [2]. If $X = (X_k) \in F[p]$, we say that $X = (X_k)$ strongly convergent to fuzzy number X_0.

Now we will give some other topological properties of these sequence spaces that we defined with the reference [14].

THEOREM 1. Let $0 < h = Inf_k\, p_k \le \sup_k\, p_k = H < \infty$. For any Orlicz function M which satisfies Δ_2-condition, we have

$$F[A_\sigma, \lambda, p] \subset F[A_\sigma, M, \lambda, p],$$

where

$$F[A_\sigma, \lambda, p] = \left\{ X = (X_k) : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} a_{mk} \left(\frac{\bar{d}|}{} \right. \right. \\ \left. 0, uniformly\ in\ m \right\}$$

.

PROOF. Let $X = (X_k) \in F[A_\sigma, \lambda, p]$ so that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} a_{mk} \left(\frac{\bar{d}\left(X_{\sigma^k(m)}, X_0\right)}{\rho} \right)^{p_k} = 0, for\ some\ \rho$$

.

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \le t \le \delta$. Write $y_{km} = \dfrac{\bar{d}\left(X_{\sigma^k(m)}, X_0\right)}{\rho}$ and consider

$$\frac{1}{\lambda_n} \sum_{k\in I_n} [M(y_{km})]^{p_k} = \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ y_{km}\ge\delta}} [M(y_{km})]^{p_k} + \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ y_{km}<\delta}} [M(y_{km})]^{p_k}$$

For the first summation above, we can write

$$\frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ y_{km}<\delta}} [M(y_{km})]^{p_k} < \lambda_n\left(\varepsilon, \varepsilon^h\right)$$

by using continuity of M. For the second summation, we will make following procedure. We have

$$y_{km} \le \frac{y_{km}}{\delta} < 1 + \frac{y_{km}}{\delta}$$

Since M is non-decreasing and convex, it follows that

$$M(y_{km}) < M\left(1 + \frac{y_{km}}{\delta}\right) \le \frac{1}{2} M(2) + \frac{1}{2} M\left(2\frac{y_{km}}{\delta}\right).$$

Since M satisfies Δ_2-condition , we can write

$$M(y_{km}) \le \frac{L}{2}\left(\frac{y_{km}}{\delta}\right) M(2) + \frac{L}{2}\left(\frac{y_{km}}{\delta}\right) M(2) = L\left(\frac{y_{km}}{\delta}\right) M(2)$$

.

So,

$$\sum_{\substack{k\in I_n \\ y_k>\delta}} [M(y_{km})]^{p_k} \le \max(1, \left[LM(2)\delta^{-1}\right]^H) \lambda_n \frac{1}{\lambda_n} \sum_{k\in I_n} [(y_{km})]^{p_k}$$

.

Then,

$$\frac{1}{\lambda_n} \sum_{k\in I_n} [M(y_{km})]^{p_k} \le \max\left(\varepsilon, \varepsilon^h\right) +$$

$$\max(1, \left[LM(2)\delta^{-1}\right]^H) \frac{1}{\lambda_n} \sum_{k\in I_n} [(y_{km})]^{p_k}$$

Taking the limit $\varepsilon \to 0$ and $n \to \infty$ uniformly in m, it follows that $X = (X_k) \in F[A_\sigma, M, \lambda, p]$.

THEOREM 2. Let $0 \le p_k \le q_k$ and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then

$$F[A_\sigma, M, \lambda, q] \subset F[A_\sigma, M, \lambda, p].$$

PROOF. Using the same technique of Theorem 3.3 of Murseleen and Basarir [2], it is easy to prove of the theorem.

Now, we give some well-known definitions:

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_o if for every $\varepsilon > 0$,

$\lim_n \frac{1}{n}\left|\left\{k \le n: \quad \bar{d}(X_k, X_o) \ge \varepsilon\right\}\right| = 0$ .It is noted that if a sequence $X = (X_k)$ of fuzzy numbers is converges to a fuzzy number X_o, then it is statistically converges to X_o. But the converse need not to be true [4].

A fuzzy sequence $X = (X_k)$ is said to be $\lambda(\sigma) -$ statistically convergent to fuzzy number X_0, if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\left|\left\{k \le n: \left[\bar{d}\left(X_{\sigma^k(m)}, X_0\right)\right] \quad \ge \epsilon\right\}\right| = ($$

,

where the vertical bars in the set indicate the number of elements in the enclosed set. We denote $S_\lambda(\sigma)$, for all statistical convergent fuzzy numbers sequences $X = (X_k)$

In the case $\sigma(n) = n$ for all n, we obtain λ -statistically convergent sequence spaces S_λ, which was defined and studied by Savas [5].

THEOREM 3. The following statements are valid:

a) $F[A_\sigma, M, \lambda, p] \subset S_\lambda(\sigma)$,

b) If $X = (X_k) \in l_\infty(F,\sigma) \cap S_\lambda(\sigma)$, then $X = (X_k) \in F[A_\sigma, M, \lambda, p]$,

c) $l_\infty(F,\sigma) \cap S_\lambda(\sigma) = l_\infty(F,\sigma) \cap F[A_\sigma, M, \lambda, p]$,

where

$$l_\infty(F,\sigma) = \left\{X = (X_k): \quad \sup_{k,m} \bar{d}\left(X_{\sigma^k(m)}, X_0\right) \right.$$

$<\infty$.

PROOF.a) Let $\varepsilon > 0$ and $X = (X_k) \in F[A_\sigma, M, \lambda, p]$. Then we have

$$\sum_{k\in I_n}\left[\bar{d}(X_k, X_o)\right]^{p_k} \ge \varepsilon^H$$

$$\left|\left\{k \in I_n: \quad \left[\bar{d}(X_k, X_o)\right]^{p_k} \ge \varepsilon\right\}\right|.$$

Hence $X = (X_k) \in S_\lambda(\sigma)$.

b) Suppose that $X = (X_k) \in S_\lambda(\sigma)$ and $X = (X_k) \in l_\infty(F,\sigma)$. Since $X = (X_k)$ is bounded, we write $\bar{d}\left(X_{\sigma^k(m)}, X_0\right) \le T$ for all k and m. Given $\varepsilon > 0$, let

$$A_n = \left\{k \in I_n: \quad \left[\bar{d}(X_k, X_o)\right]^{p_k} \ge \varepsilon\right\} \text{ and}$$

$$B_n = \left\{k \in I_n: \quad \left[\bar{d}(X_k, X_o)\right]^{p_k} < \varepsilon\right\}.$$

Then we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[\bar{d}(X_k, X_o)\right]^{p_k} = \frac{1}{\lambda_n}\sum_{k\in A_n}\left[\bar{d}(X_k, X_o)\right]^{p_k} + \frac{1}{\lambda_n}\sum_{k\in B_n}\left[\bar{d}(X_k, X\right.$$

$$\le \frac{T}{\lambda_n}|A_n| + \varepsilon^H .$$

Hence $X = (X_k) \in F[\lambda, p]$.

c) This proof follows from (a) and (b).

THEOREM 4. If $\lim Inf_n \frac{\lambda_n}{n} > 0$, then $S(\sigma) \subseteq S_\lambda(\sigma)$,

where

$$S(\sigma) = \left\{X = (X_k): \lim_n \frac{1}{n}\left|\left\{k \in I_n: \quad \left[\bar{d}(X_k, X_o)\right]^{p_k} \ge \varepsilon\right\}\right| = 0\right\}$$

.

PROOF. Let $X = (X_k) \in S(\sigma)$. For given $\varepsilon > 0$, we get

$$\left\{k \le n: \quad \left[\bar{d}(X_k, X_o)\right]^{p_k} \ge \varepsilon\right\} \supset A_n$$

where A_n is as is in Theorem 5. Thus,

$$\frac{1}{n}\left|\left\{k \le n : \quad \left[\bar{d}\left(X_k, X_o\right)\right]^{p_k} \ge \varepsilon\right\}\right| \ge \frac{1}{n}\left|A_n\right| = \frac{\lambda_n}{n} \cdot \frac{1}{\lambda_n}\left|A_n\right|$$

.

Taking limit as $n \to \infty$ and using $\lim Inf_n \frac{\lambda_n}{n} > 0$, we get $X = (X_k) \in S_\lambda(\sigma)$.

CONCLUSION

In this conference paper, we studied some sequence spaces of fuzzy numbers defined by Orlicz function. Statistical convergence and some inclusion relations were argued.

REFERENCES

[1] Diamond,P.and Kloeden P., “Metric spaces of fuzzy sets”, Fuzzy Sets and Systems, 35(1990), 241-249.

[2] Mursaleen and Basarir,M., “On some new sequence spaces of fuzzy numbers”,Indian J.pure appl.Math., 34(9),(2003),1351-1357.

[3] Nanda, S., “On sequences of fuzzy numbers”, Fuzzy Sets and Systems, 33(1989), 123-126.

[4] Nuray;F.and Savas,E.,”Statistical convergence of sequences of fuzzy numbers”, Math.Slovaca, 45(1995), 269-273.

[5] Savas,E.,”On strongly λ -summable sequences of fuzzy numbers”, Information Sciences, 125(2000), 181-186.

[6] Zadeh,L.A.,”Fuzzy sets”, Inform Control,8(1965), 338-353.

[7] Matloka,M.,”Sequences of fuzzy numbers”, BUSEFAL, 28(1986), 28-37.

[8] Lindenstrauss, J. And Tzafriri,L.”On Orlicz sequence spaces”, Israel J.Math.10(3) (1971), 379-390.

[9] Parashar,S.D. and Choundhary,B.,”Sequence spaces defined by Orlicz functions”, Indian J.pure appl.Math., 25(14),(1994), 419-428.

[10] Esi,A.,Isık,M. and Esi,A.,”On some new sequence spaces defined by Orlicz functions”, Indian J.pure appl.Math., 35(1),(2004), 31-36.

[11] Esi,A.,”Some new sequence spaces defined by Orlicz functions”,Bulletin of The Institute of Mathematics,Academia Sinica,27(1)(1999), 71-76.

[12] Esi,A. and Et,M.,”Some new sequence spaces defined by a sequence of Orlicz functions” Indian J.pure appl.Math., 31(8)(2000), 967-972.

[13] Esi,Ayhan, *On some new paranormed sequence spaces of fuzzy numbers defined by Orlicz functions and statistical convergence*, Mathematical Modelling and Analysis, 1(4), 379-388, 2006.

[14] Esi,Ayhan, Some new generalized classes of sequences of fuzzy numbers defined by a Orlicz function-I, submitted to 1.st International Conference on Modern and Advanced Researc,, ICMAR, 2023.