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# A Note on Relation Between Compositions and Two-Variable Polynomials

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***Abstract*** **–** In this study, we expressed the composition set of a positive integer with the help of set theory by expressing the partition and composition of positive integers. We have defined a product function for compositions of a positive integer. In addition, the correlation between the composition of a positive integer and the polynomial in the two-variable was obtained with the help of the product function defined by expressing the polynomial in the generalized two-variable.

*Keywords – Compositions of an Integer, Two-Variable Polynomials, Recurrence Relation, Sets of Compositions of an Integer, Partitions of an Integer.*

## I. INTRODUCTION

Partition of positive integers has been the focus of attention from past to present. Partition theory arose with Leibniz's question of how many ways a positive integer can be written as the sum of positive integers and after this question, many studies have been done on the partition of numbers ([4],[6],[7],[9],[11]). The number of partitions of a positive integer n is the number of ways $n$ can be written as the sum of positive integers.

Partition is divided into compositions and partition. It is important that the totals do not change in the composition. In partition, it is not important that the sums are commutative.

For the positive integer n, the partition number is denoted by p(n), while the composition number is P(n) notation.

Example: The number 5 has 7 partitions and the number of compositions is 16.

- The set of partition of 5 is {5; (1+4); (2+3); (1+1+3); (2+2+1); (2+1+1+1); (1+1+1+1+1)} and 5 has 7 partitions.
- The set of composition of 5 is {5; (1+4); (4+1); (2+3); (3+2); (1+1+3); (1 +3+1); (3 +1+1); (2+2+1); (1+2+2); (2+1+2); (2+1+1+1); (1+2+1+1); (1+1+2+1); (1+1+1+2); (1+1+1+1+1)} and 5 has 16 compositions.

Euler investigated the generating function of the number of partitions of an integer $n$; as follows

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)\, x^n$$

where 0<x<1 *[Apostol, 1976]*.

Gupta also expressed the composition number of n positive integers as

$$P(n) = 2^{n-1}$$

*[Gupta, 1970]*.

Since in this study, generalized two-variable polynomials will be associated with the compositions of an integer, it is useful to express generalized two-variable polynomials.

Definition: The generating function for the generalized two-variable polynomials $G_j(t,y;k,m,n)$,

$$H(x,t,y;k,m,n)=\frac{1}{1-t^k x-y^m x^{m+n}} = \sum_{j=0}^{\infty} G_j(t,y;k,m,n)x^j,$$

where $m,n,k\in\mathbb{N}$, $x,y\in\mathbb{R}$ and $t\in\mathbb{C}$ *[Ozdemir and Simsek, 2016]*.

In [11], Ozdemir and Simsek give explicit formula for the polynomials $G_j(x,y;k,m,n)$ by the following theorem:

Theorem: Let $m,n,k\in\mathbb{N}$, $x,y\in\mathbb{R}$ and $t\in\mathbb{C}$

$$G_j(t,y;k,m,n) = \sum_{c=0}^{\left[\!\left[\frac{j}{m+n}\right]\!\right]} \binom{j-c(m+n-1)}{c} y^{mc}x^{jk-mck-nck},$$

where $[\![a]\!]$ is the largest integer $\leq a$.

## II. Materials and Method

We remember some phrase from [1]. Let n be a positive integer and we define the set

$$P_n=\{(a_1,a_2,\dots,a_t): a_1+a_2+\cdots+a_t = n,\quad a_i,t\in\mathbb{Z}^+\}.$$

In [1], we have reared the set $P_{n+1}$ of composition for a positive integer $n$ by using recurrence relations on the set $P_n$. First, we recall operations with a partition $a=(a_1,a_2,\dots,a_t)$ of integer $n$;

$$1\odot a=(1,a_1,a_2,\dots,a_t)$$
$$1\oplus a=(1+a_1,a_2,\dots,a_t)$$

Then $1\odot a, 1\oplus a\in P_{n+1}$ and so we also use the notaions $1\odot P_n, 1\oplus P_n$ for the set of new type elements, i.e.

$$1\odot P_n=\{1\odot a : a\in P_n\},$$
$$1\oplus P_n=\{1\oplus a : a\in P_n\}.$$

In [1], by using the composition partition set of an integer n, we define the notation $\bar{a}=a_1.a_2.\dots.a_t$ for multiplication of summand where $n=a_1+a_2+\cdots+a_t$. The sum of multiplication of summand in the composition set $P_n$ define the function from the composition sets of integers to positive integers defined by

$$T(P_n)=T_n=\sum_{a\in P_n}\bar{a}.$$

We may assume that $T_0=1$ and $T_n=T(P_n)$ is defined the multiplication sum of the composition set $P_n$ (or the multiplication sum of the integer n).

Example: For n=4, we have

$P_4=\{(4),(3,1),(1,3),(2,2),(2,1,1),(1,2,1),(1,1,2),(1,1,1,1)\}$ and

$$T_4=\mathrm{T}(P_4)=4+3.1+1.3+2.2+2.1.1+1.2.1+1.1.2+1.1.1.1=21.$$

Now the relation of generalized two-variable polynomials with $T_n$ will be shown.

Lemma: Let $m,n,k\in\mathbb{N}$, $t,y\in\mathbb{R}$ and $x\in\mathbb{C}$

$$T_n=G_{n-1}(3,-1;1,1,1).$$

Proof: Let $t=3, k=1, m=1, y=-1$ and $n=1$ in the

$$\frac{1}{1-t^k x-y^m x^{m+n}}=\sum_{j=0}^{\infty} G_j(t,y;k,m,n)x^j$$

generating function given in [11],

$$\frac{1}{1-3x+x^2}=\sum_{j=0}^{\infty} G_j(3,-1;1,1,1)x^j$$

is obtained. Then we have

$$\sum_{j=0}^{\infty} T_j x^j=\frac{x}{1-3x+x^2} = x\sum_{j=0}^{\infty} G_j(3,-1;1,1,1)x^j.$$

Hence, for $n\geq 1$, we have that

$$T_n = G_{n-1}(3, -1; 1,1,1).$$

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