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Generating Function For the Number of Summand Size Restricted Compositions

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Abstract – In this paper, we explored constrained partition with the help of previously defined notations and sets for partition and compositions. We defined clusters for the summand size restricted compositions. Finally, we obtained the generating function for the number of summand size restricted compositions.

Keywords – Compositions of an Integer, Partitions of an Integer, Recurrence Relation, Sets of Compositions of an Integer, Restricted Compositions.

I. INTRODUCTION

The foundations of the partition theory date back to 1674. "How many ways can a positive number be written as the sum of positive numbers?" With the question of the partition theory, work has begun. From the past to the present, many scientists have worked on the partition of positive integers ([6],[7],[8],[9],[11],[12],[13],[16],[17],[22]). For a positive integer *n*, the partition function to be studied is the number of ways *n* can be written as a sum of positive integer *n*. The summands are called parts.

Partitions are divided into compositions and partition. The displacement of summands in partition is not important.

For the positive integer n, the partition number is denoted by *p(n)*, while the composition number is denoted by *P(n)*.

Example: The number 6 has 11 partitions, and the number of compositions is 32.

- The set of partition of 6 is {6; (1,5); (2,4); (1,1,4); (3,3); (3,2,1); (3,1,1,1); (2,2,2); (2,2,1,1); (2,1,1,1,1); (1,1,1,1,1,1)} and 6 has 11 partitions.
- The set of composition of 6 is {6; (1,5); (5,1); (2,4); (4,2); (1,1,4); (1 ,4,1); (4 ,1,1); (3,3); (3,2,1); (3,1,2); (2,3,1); (2,1,3); (1,2,3); (1,3,2); (3,1,1,1); (1,3,1,1); (1,1,3,1); (1,1,1,3); (2,2,2); (2,2,1,1); (2,1,2,1); (2,1,1,2); (1,2,2,1); (1,1,2,2); (1,2,1,2); (1,1,1,1,2); (1,1,1,2,1); (1,1,2,1,1); (1,2,1,1,1); (2,1,1,1,1); (1,1,1,1,1,1)} and 6 has 32 compositions.

Euler investigated the generating function of the number of partitions of an integer n; as follows

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)\, x^n$$

where 0<x<1 *[Apostol, 1976]*.

The number of composition of an positive integers n as

$$P(n) = 2^{n-1}$$

[Gupta, 1970 ve Sılls 2011].

As the works on the partition theory improved, new information was obtained by restricted partition. In the literature, the restricted partitions are significant as unrestricted partition of an integer ([7], [10], [18]).

From [10, page 309], we recall that the number of partitions of k into parts not exceeding m is denoted by $p_m(k)$ for integers m, k. Then $p_m(k) = p(k)$ for $m \geq k$. It is clear that $p_m(k)$ is less than $p(k)$and the computation of $p_m(k)$ is easier for integers m, k. The generating function for the number of partitions of k into parts not exceeding m is defined as

$$F_m(x) = \prod_{i=1}^{m} \frac{1}{1-x^i} = 1 + \sum_{i=0}^{\infty} p_m(i)\, x^i.$$

II. Materials and Method

In this note we focus on the summand size restricted compositions.

We remember some phrase from [1]. Let n be a positive integer and we define the set

$$P_n = \{(a_1, a_2, \dots, a_t) : a_1 + a_2 + \cdots + a_t = n, \quad a_i, t \in \mathbb{Z}^+\}.$$

In [1], we have reared the set P_{n+1} of composition for a positive integer n by using recurrence relations on the set P_n. First, we recall operations with a partition $a = (a_1, a_2, \dots, a_t)$ of integer n;

$$1 \odot a = (1, a_1, a_2, \dots, a_t)$$
$$1 \oplus a = (1 + a_1, a_2, \dots, a_t)$$

Then $1 \odot a, 1 \oplus a \in P_{n+1}$ and so we also use the notaions $1 \odot P_n$, $1 \oplus P_n$ for the set of new type elements, i.e.

$$1 \odot P_n = \{1 \odot a : a \in P_n\},$$
$$1 \oplus P_n = \{1 \oplus a : a \in P_n\}.$$

Now we recall some expression from [4].

Let n, a being positive integers. $P_{n,a}$ is the set of composition of *n* with restriction a.

$$P_{n,a} = \{(x_1, x, \dots, x_m) : x_1 + x_2 + \cdots + x_m = n, \; a \geq x_i, \forall i \in \{1,2, \dots, m\}\}.$$

Therefore $|P_{n,a}|$ is number of compositions of *n* with restriction a.

Example: For compositions of *6* with restriction *2*, $P_{6,2}$= (2,2,2); (2,2,1,1); (2,1,2,1); (2,1,1,2); (1,2,2,1); (1,1,2,2); (1,2,1,2); (1,1,1,1,2); (1,1,1,2,1); (1,1,2,1,1); (1,2,1,1,1); (2,1,1,1,1); (1,1,1,1,1,1)} and $|P_{6,2}|$=13.

Now we get the generating function for the numbers of a positive integers with restriction integers a.

Theorem: Let a is a positive integer. The generating function for the numbers of a positive integers with restriction integers a is that

$$\sum_{n=0}^{\infty} |P_{n,a}| x^n = \frac{1-x}{1-2x+x^{a+1}}.$$

Proof: Let a is a positive integer.

$$\sum_{n=0}^{\infty} |P_{n,a}| x^n = \sum_{n=0}^{a-1} |P_{n,a}| x^n + |P_{a,a}| x^a + x \sum_{n=a}^{\infty} |P_{n+1,a}| x^n .$$

We recall $|P_{n+1,a}| = 2|P_{n,a}| - |P_{n-a,a}|$ in [4]. Then using the recurrence, we rewrite the series

$$\begin{aligned}
\sum_{n=0}^{\infty} |P_{n,a}| x^n &= \sum_{n=0}^{a-1} |P_{n,a}| x^n + |P_{a,a}| x^a + x \sum_{n=a}^{\infty} \left(2|P_{n,a}| - |P_{n-a,a}|\right) x^n \\
&= \sum_{n=0}^{a-1} |P_{n,a}| x^n + |P_{a,a}| x^a + 2x \sum_{n=a}^{\infty} |P_{n,a}| x^n - x \sum_{n=a}^{\infty} |P_{n-a,a}| x^n \\
&= \sum_{n=0}^{a-1} |P_{n,a}| x^n + |P_{a,a}| x^a + 2x \sum_{n=a}^{\infty} |P_{n,a}| x^n - x \sum_{n=a}^{\infty} |P_{n-a,a}| x^n \\
&\quad + 2x \sum_{n=0}^{a-1} |P_{n,a}| x^n - 2x \sum_{n=0}^{a-1} |P_{n,a}| x^n .
\end{aligned}$$

Then we get

$$\sum_{n=0}^{\infty}|P_{n,a}|x^n = \sum_{n=0}^{a-1}|P_{n,a}|x^n + |P_{a,a}|x^a$$
$$+ 2x\sum_{n=0}^{\infty}|P_{n,a}|x^n - x\sum_{n=a}^{\infty}|P_{n-a,a}|x^n$$
$$- 2x\sum_{n=0}^{a-1}|P_{n,a}|x^n .$$

Hence

$$\sum_{n=0}^{\infty}|P_{n,a}|x^n = \frac{(1-2x)\sum_{n=0}^{a-1}|P_{n,a}|x^n + |P_{a,a}|x^a}{1-2x+x^{a+1}}$$
$$= \frac{(1-2x)(1+\sum_{n=1}^{a-1}2^{n-1}x^n) + 2^{a-1}x^a}{1-2x+x^{a+1}}$$
$$= \frac{(1-2x)(1+x\sum_{n=1}^{a-1}(2x)^{n-1}) + 2^{a-1}x^a}{1-2x+x^{a+1}}.$$

By doing the necessary arithmetic operations, the following expression is obtained:

$$\sum_{n=0}^{\infty}|P_{n,a}|x^n = \frac{(1-2x)\left(1+x\frac{1-(2x)^{a-1}}{1-2x}\right)+2^{a-1}x^a}{1-2x+x^{a+1}} =$$
$$\frac{\left((1-2x)+x(1-2x)\frac{1-(2x)^{a-1}}{1-2x}\right)+2^{a-1}x^a}{1-2x+x^{a+1}} =$$
$$\frac{(1-2x)+x(1-(2x)^{a-1})+2^{a-1}x^a}{1-2x+x^{a+1}} =$$
$$\frac{(1-2x)+x-2^{a-1}x^a+2^{a-1}x^a}{1-2x+x^{a+1}}.$$

Thus, the proof is completed.

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