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Applications of Alughte Transform for Berezin Radius Inequalities

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Abstract – In functional analysis, linear operators induced by functions are frequently encountered; these contain Hankel operators, constitution operators, and Toeplitz operators. The symbol of the resultant operator is another name for the inciting function. In many instances, a linear operator on a Hilbert space $\mathcal{H}$ results in a function on a subset of a topological space. As a result, we regularly investigate operators induced by functions, and we may also investigate functions induced by operators. The Berezin sign is a wonderful representation of an operator-function relationship. F. Berezin proposed the Berezin switch in [8], and it has proven to be a vital tool in operator theory given that it utilizes many essential aspects of significant operators. Many mathematicians and physicists are fascinated by the Berezin symbol of an operator defined on the functional Hilbert space. The Berezin radius inequality has been extensively studied in this situation by a number of mathematicians. In this paper, we use the Alughte transform and the generalized Alughte transform to develop Berezin radius inequalities for Hilbert space operators. We additionally offer fresh Berezin radius inequality results. Huban et al. [15] and Başaran et al. [6] supply the Berezin radius inequality.

Keywords – Berezin Symbol, Functional Hilbert Space, Alughte Transform, Generalized Alughte Transform

I. INTRODUCTION

Hankel operators, constitution operators, and Toeplitz operators are some examples of the linear operators induced by functions that are commonly seen in functional analysis. The symbol of the resultant operator is another name for the inciting function. In many instances, a linear operator on a Hilbert space $\mathcal{H}$ results in a function on a subset of a topological space. As a result, we frequently look into operators that functions induce, and occasionally we look into functions that operators induce. The Berezin sign is a wonderful representation of an operator-function relationship. F. Berezin proposed the Berezin switch in [8], and it has proven to be a vital tool in operator theory given that it utilizes many essential aspects of significant operators. The Berezin symbol of an operator defined on the reproducing kernel Hilbert space fascinates many mathematicians and physicists. The Berezin radius inequality has been extensively studied in this situation by a number of mathematicians. In this paper, we use the Alughte transform and the generalized Alughte transform to develop Berezin radius inequalities for Hilbert space operators. We additionally offer fresh Berezin radius inequality results. Huban et al. and Başaran et al. supply the Berezin radius inequality.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ define the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Recall the functional Hilbert space (briefly, FHS) $\mathcal{H}=\mathcal{H}(\mathfrak{X})$ is a Hilbert space on some set (nonempty) F, such that evalution functional $\psi_\varsigma(f), \varsigma \in$ F, are continouns on a $\mathcal{H}$. Hence, by Riesz representation theorem, for each $\varsigma \in \mathfrak{X}$, there is an unique element $k_\varsigma \in \mathcal{H}$ such that $f(\varsigma) = \langle f, k_\varsigma \rangle$, for all $f \in \mathcal{H}$. The family $\{k_\varsigma : \varsigma \in \mathfrak{X}\}$ is called the reproducing kernel in $\mathcal{H}$. For $\varsigma \in$ F, $\widehat{k_\varsigma} = \frac{k_\varsigma}{\|k_\varsigma\|}$ is defined the normalized reproducing kernel.

For $V \in \mathbb{B}(\mathcal{H})$, the function $\tilde{V}$ defined on $\mathfrak{X}$ by $\tilde{V}(\varsigma) = \langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle$ is the Berezin symbol of V. Berezin symbol firstly has been introduced by Berezin ([8]). The Berezin set and Berezin number of the operator V are defined by
$Ber(V) = \{\tilde{V}(\varsigma): \varsigma \in \mathfrak{X}\}$
and
$ber(V) = sup\{\tilde{V}(\varsigma): \varsigma \in \mathfrak{X}\} = sup_{\varsigma\in F}\left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|$
respectively (see, [18, 19]). The Berezin symbol has been thoroughly studied for the Toeplitz and Hankel operators on the Hardy and Bergman spaces. It is frequently used in many fields of study and uniquely identifies an operator. We recommend the reader to [4-7, 13] for more information on the Berezin symbol.

In a FHS, the Berezin range and Berezin number of an operator V are a subset of numerical range and numerical radius of V, respectively. There are interesting properties of numerical range and numerical radius. For basic properties numerical radius, we refer to [10, 11, 14]. The fact that

$$ber(V) \leq w(V) \leq \|V\| \quad (1.1)$$

is significant. It is common knowledge that for all $V \in \mathbb{B}(\mathcal{H})$,

$$ber(V) \leq \frac{1}{2}\left(\|V\|_{ber} + \|V^2\|_{ber}^{1/2}\right), \quad (1.2)$$

(see [16, 17]). Huban et al. consolidationed the second inequality in (1.2) using the Cartesian decomposition for operators in [15]:

$$\frac{1}{4}\|V^*V + VV^*\|_{ber} \leq ber^2(V) \leq \frac{1}{2}\|V^*V + VV^*\|_{ber} \quad (1.3)$$

for any operator $V \in \mathbb{B}(\mathcal{H})$. The same authors have also obtained that

$$\left(ber(V)\right)^\varepsilon \leq \frac{1}{2}\left\||V|^{2\gamma\varepsilon} + |V^*|^{2(1-\gamma)\varepsilon}\right\|_{ber}, \quad (1.4)$$

and

$$\left(ber(V)\right)^\varepsilon \leq \frac{1}{2}\|\gamma|V|^{2\varepsilon} + (1-\gamma)|V^*|^{2\varepsilon}\|_{ber}, \quad (1.5)$$

for $0 < \gamma < 1$ and $\varepsilon \geq 1$ (see, [16, Th. 3.1 and Th. 3.2]. They also showed the following as stronger than [3]:

$$ber(V) \leq \frac{1}{2}\||V| + |V^*|\|_{ber} \quad (1.6).$$

Another improvement has been established by Başaran et al. [6] and Huban et al. [15]:

$$ber^2(X) \leq \frac{1}{2}\||V|^2 + |V^*|^2\|_{ber}, \quad (1.7).$$

Let $T = U|T|$ be the polar decomposition of the bounded linear operator T with U a partial isometry. The Aluthge transform $T^\sim$ of T is denoted by $T^\sim = |T|^{1/2}U|T|^{1/2}$, see [1].

For $t \in (0,1)$

$$\Delta_t(T) = |T|^t U |T|^{1-t}$$

is called the generalized Aluthge transform (see, [2, 9]).

II. MAIN RESULT

Before we start the section, we presented two lemmas.

Lemma 2.1. $A, B \in \mathbb{B}(\mathcal{H})$ and $\rho, \sigma > 1$ with $\frac{1}{\rho} + \frac{1}{\sigma} = 1$. Then

(i) If $\varepsilon > 0$ and $\rho\varepsilon, \sigma\varepsilon \geq 2$, then

$$ber^{2\varepsilon}(AB) \leq \frac{1}{2}\left\|\frac{1}{\rho}|A|^{2\rho\varepsilon} + \frac{1}{\sigma}|B|^{2\varepsilon\sigma}\right\|_{ber} \quad (2.1)$$

(ii) If $\varepsilon \geq 1$ and $\rho\varepsilon, \sigma\varepsilon \geq 2$, then

$$ber^{2\varepsilon}(AB) \leq \frac{1}{2}\left[\|A\|_{Ber}^{2\varepsilon}\|B\|_{Ber}^{2\varepsilon} + ber^\varepsilon(|B|^2|A^*|^2)\right] \quad (2.2)$$

(iii) If $\varepsilon \geq 1$ and $\rho\varepsilon, \sigma\varepsilon \geq 2$, then

$$ber^{2\varepsilon}(AB) \leq \frac{1}{2}\left\|\frac{1}{\rho}|A|^{2\rho\varepsilon} + \frac{1}{\sigma}|B|^{2\varepsilon\sigma}\right\|_{ber} + ber^\varepsilon(|B|^2|A^*|^2) \quad (2.3)$$

(see, [12]).

Lemma 2.2. $A, B \in \mathbb{B}(\mathcal{H})$, then for $\gamma\epsilon[0,1]$ and $\varepsilon \geq 1$,

$$ber^2(AB) \leq \frac{1}{2}\|(1-\gamma)|A^*|^{2\varepsilon} + \gamma|B|^{2\varepsilon}\|_{ber}^{1/\varepsilon}\|A\|_{ber}^{2\gamma}\|B\|_{ber}^{2(1-\gamma)} \quad (2.4)$$

and

$ber^2(AB) \le \|(1-\gamma)|A^*|^{2\varepsilon}$
$+\gamma|B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}\|(1-\gamma)|A^*|^{2\varepsilon}$
$+\gamma|B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}.$

In particular,

$ber^2(AB)$
$\le \frac{1}{2^{1/\varepsilon}}\||A^*|^{2\varepsilon}+|B|^{2\varepsilon}\|_{ber}^{1/\varepsilon}\|A\|_{Ber}\|B\|_{Ber}$

and

$$ber(AB) \le \frac{1}{2^{1/\varepsilon}}\||A^*|^{2\varepsilon}+|B|^{2\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}$$

(see, [12])

Now, we get the following inequalities for one operator:

Theorem 2.3. Let $\mathcal{H}=\mathcal{H}(\mathfrak{X})$ be a FHS. Let $A \in \mathbb{B}(\mathcal{H})$ and $\gamma \epsilon [0,1]$, then

$ber^{2\varepsilon}(A)$
$\le \left\|\frac{1}{\rho}|A^*|^{2\gamma\varepsilon\rho}+\frac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber}, \qquad (2.5)$

for $\varepsilon>0$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\ge 1$. Also, for $\varepsilon \ge 1$

$ber^{2\varepsilon}(A)$
$\le \frac{1}{2}\left[\|A\|_{Ber}^{2\varepsilon}+ber^{\varepsilon}\left(|A|^{2(1-\gamma)}|A^*|^{2\gamma}\right)\right]. \qquad (2.6)$

If $\varepsilon \ge 1$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\ge 2$, then

$ber^{2\varepsilon}(A) \le \frac{1}{2}\left(\left\|\frac{1}{\rho}|A^*|^{2\gamma\varepsilon\rho}+\frac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber}+\right.$
$\left. ber^{\varepsilon}\left(|A|^{2(1-\gamma)}|A^*|^{2\gamma}\right)\right) \qquad (2.7)$

Moreover, if $\gamma\epsilon[0,1]$ and $\varepsilon \ge 1$, then

$ber^2(A) \le \left\|(1-\delta)|A^*|^{2\gamma\varepsilon}+\right.$
$\left.\delta|A|^{2(1-\gamma)\varepsilon}\right\|_{ber}^{1/\varepsilon}\|A\|_{ber}^{2[\gamma\delta+(1-\delta)(1-\gamma)]}. \qquad (2.8)$

Proof. If we write $T=U|A|^{\gamma}$ and $S=U|A|^{(1-\gamma)}$ in (2.1) and can see that $TS=U|A|=A$,

$|T^*|^2=TT^*=U|A|^{\gamma}|A|^{\gamma}U^*=U|A|^{2\gamma}U^*$
$=|A^*|^{2\gamma},$

then

$$ber^{2\varepsilon}(A) \le \frac{1}{2}\left\|\frac{1}{\rho}|A^*|^{2\gamma\varepsilon\rho}+\frac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber}$$

which proves (2.5).

The same choice T and S in (2.2) has

$ber^{2\varepsilon}(A)$
$\le \frac{1}{2}\left[\|U|A|^{\gamma}\|_{Ber}^{2\varepsilon}+ber^{\varepsilon}\left(|A|^{2(1-\gamma)}|A^*|^{2\gamma}\right)\right]. \qquad (2.9)$

It is clear that

$$|T|^2=T^*T=|A|^{\gamma}U^*U|A|^{\gamma}=|A|^{\gamma}|A|^{\gamma}=|A|^{2\gamma}$$

since U is an isometry on $ran(T)$. Then $\|U|A|^{\gamma}\|_{ber}^{2\varepsilon}=\|A\|_{ber}^{2\gamma\varepsilon}$ and by (2.9) we have (2.6).

If $\varepsilon \ge 1$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\ge 2$, then by (2.3) we have (2.7).

Moreover, if we use (2.4) for $T=U|A|^{\gamma}$ and $S=U|A|^{(1-\gamma)}$, then we have for $\delta\epsilon[0,1]$ and $\varepsilon\ge 1$ that

$ber^2(A)$
$\le \frac{1}{2}\|(1-\delta)|A^*|^{2\gamma\varepsilon}$
$+\delta|A|^{2(1-\gamma)\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}\|A\|_{ber}^{2(1-\delta)(1-\gamma)}\|A\|_{ber}^{2\gamma\delta}$
$\le \frac{1}{2}\|(1-\delta)|A^*|^{2\gamma\varepsilon}$
$+\delta|A|^{2(1-\gamma)\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}\|A\|_{ber}^{2[\gamma\delta+(1-\delta)(1-\gamma)]}$

which proves (2.8).

Corollary 2.4. Let $A\in\mathbb{B}(\mathcal{H})$. Then

$ber^{2\varepsilon}(A) \le \left\|\frac{1}{\rho}|A^*|^{\varepsilon\rho}+\frac{1}{\sigma}|A|^{\sigma\varepsilon}\right\|_{ber}, \qquad (2.10)$

for $\varepsilon>0$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\ge 1$. Also, for $\varepsilon\ge 1$

$ber^{2\varepsilon}(A) \le \frac{1}{2}[\|A\|_{Ber}^{2\varepsilon}+ber^{\varepsilon}(|A||A^*|)]. \qquad (2.11)$

If $\varepsilon \ge 1$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\ge 2$, then

$ber^{2\varepsilon}(A)$
$\le \frac{1}{2}\left(\left\|\frac{1}{\rho}|A^*|^{\varepsilon\rho}+\frac{1}{\sigma}|A|^{\sigma\varepsilon}\right\|_{ber}+ber^{\varepsilon}(|A||A^*|)\right)$
(2.12)

Moreover, if $\gamma\epsilon[0,1]$ and $\varepsilon\ge 1$, then

$ber^2(A)$
$\le \|(1-\delta)|A^*|^{2\varepsilon}+\delta|A|^{\varepsilon}\|_{ber}^{1/\varepsilon}\|A\|_{ber}. \qquad (2.13)$

Remark 2.5. If we choose $\varepsilon = 1$ in (2.10), then we have

$$ber^2(A) \le \left\|\frac{1}{\rho}|A^*|^\rho + \frac{1}{\sigma}|A|^\sigma\right\|_{ber},$$

for $\rho, \sigma > 1$ with $\frac{1}{\rho}+\frac{1}{\sigma} = 1$. In particular, $\rho = \sigma = 2$ we have

$$ber^{2\varepsilon}(A) \le \frac{1}{2}\||A^*|^2 + |A|^2\|_{ber}.$$

If we write $\varepsilon = 1$ in (2.7), then we have

$$ber^2(A) \le \tfrac{1}{2}\left(\|A\|_{ber}^2 + ber^\varepsilon\left(|A|^{2(1-\gamma)}|A^*|^{2\gamma}\right)\right).$$

If we choose $\varepsilon = 1$ and $\rho = \sigma = 2$, then by (2.12) we have

$$ber^2(A) \le \frac{1}{2}\left(\||A^*|^2 + |A|^2\|_{ber} + ber(|A||A^*|)\right)$$
$$= \tfrac{1}{4}\||A^*|^2 + |A|^2\|_{ber} + \tfrac{1}{2}ber(|A||A^*|).$$

If we choose $\varepsilon = 2$ and $\rho, \sigma > 1$ with $\frac{1}{\rho}+\frac{1}{\sigma} = 1$ in (2.12), then we reach

$$ber^4(A)$$
$$\le \tfrac{1}{2}\left(\left\|\tfrac{1}{\rho}|A^*|^{2\rho} + \tfrac{1}{\sigma}|A|^{2\sigma}\right\|_{ber} + ber^2(|A||A^*|)\right),$$

which for $\rho = \sigma = 2$ has

$$ber^4(A) \le \tfrac{1}{4}\||A^*|^4 + |A|^4\|_{ber} + \tfrac{1}{2}ber^2(|A||A^*|).$$

We also get:

Theorem 2.6. Let $\mathcal{H} = \mathcal{H}(\mathfrak{X})$ be a FHS. Let $A \in \mathbb{B}(\mathcal{H})$ and $\gamma\epsilon[0,1]$, then

$$ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)$$
$$\le \left\|\tfrac{1}{\rho}|U^*|A|^\gamma|^{2\rho\varepsilon} + \tfrac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber}, \qquad (2.14)$$

for $\varepsilon > 0$, $\rho, \sigma > 1$ with $\frac{1}{\rho}+\frac{1}{\sigma} = 1$ and $\rho\varepsilon, \sigma\varepsilon \ge 1$. Also,

$$ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)$$
$$\le \tfrac{1}{2}[\|A\|_{Ber}^{2\varepsilon} + ber^\varepsilon(|A|^{2-\gamma}UU^*|A^*|^\gamma)] \qquad (2.13)$$

for $\varepsilon \ge 1$. Moreover

$$ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)$$
$$\le \tfrac{1}{2}\left(\left\|\tfrac{1}{\rho}|U^*|A|^\gamma|^{2\rho\varepsilon} + \tfrac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber} +\right.$$
$$\left. ber^\varepsilon(|A|^{2-\gamma}UU^*|A^*|^\gamma)\right) \qquad (2.14)$$

for $\varepsilon \ge 1$, $\rho, \sigma > 1$ with $\frac{1}{\rho}+\frac{1}{\sigma} = 1$ and $\rho\varepsilon, \sigma\varepsilon \ge 2$. Also, for $\delta\epsilon[0,1]$ and $\varepsilon \ge 1$, then

$$ber^2\left(\Delta_\gamma(A)\right)$$
$$\le (1-\delta)\||U^*|A|^\gamma|^{2\rho\varepsilon} +$$
$$\delta|A|^{2(1-\gamma)\sigma\varepsilon}\|_{ber}^{1/\varepsilon}\|A\|_{ber}^{2[\gamma\delta+(1-\delta)(1-\gamma)]}. \qquad (2.15)$$

Proof. If we put $\mathrm{T} = |\mathrm{A}|^\gamma\mathrm{U}$ and $\mathrm{S} = |\mathrm{A}|^{1-\gamma}$ in (2.1) and can see that $\mathrm{TS} = |\mathrm{A}|^\gamma\mathrm{U}|\mathrm{A}|^{1-\gamma} = \Delta_\gamma(\mathrm{A})$, then we have

$$ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)$$
$$\le \left\|\tfrac{1}{\rho}|U^*|A|^\gamma|^{2\rho\varepsilon} + \tfrac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber}.$$

With the same choice and by (2.2) we reach

$$ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)$$
$$\le \frac{1}{2}[\||\mathrm{A}|^\gamma\mathrm{U}\,\|_{Ber}^{2\varepsilon}\||\mathrm{A}|^{1-\gamma}\|_{Ber}^{2\varepsilon}$$
$$+ ber^\varepsilon(|\mathrm{A}|^{1-\gamma}|\mathrm{A}|^\gamma UU^*|\mathrm{A}|^\gamma)]$$
$$= \frac{1}{2}[\||\mathrm{A}|^\gamma\mathrm{U}\,\|_{Ber}^{2\varepsilon}\||\mathrm{A}|^{1-\gamma}\|_{Ber}^{2\varepsilon}$$
$$+ ber^\varepsilon(|\mathrm{A}|^{2-\gamma}UU^*|\mathrm{A}|^\gamma)]$$
$$\le \frac{1}{2}\Big[\|\mathrm{A}\,\|_{Ber}^{2\gamma\varepsilon}\|\mathrm{A}\|_{Ber}^{2(1-\gamma)\varepsilon}$$
$$+ ber^\varepsilon(|\mathrm{A}|^{2-\gamma}UU^*|\mathrm{A}|^\gamma)\Big]$$
$$= \frac{1}{2}[\|\mathrm{A}\,\|_{Ber}^{2\varepsilon}$$
$$+ ber^\varepsilon(|\mathrm{A}|^{2-\gamma}UU^*|\mathrm{A}|^\gamma)]$$

which proves (2.13).

If $\varepsilon \ge 1$, $\rho, \sigma \ge 1$ with $\frac{1}{\rho}+\frac{1}{\sigma} = 1$ and $\rho\varepsilon, \sigma\varepsilon \ge 2$, the from (2.3) we have (2.14).

If we take $\mathrm{T} = |\mathrm{A}|^\gamma\mathrm{U}$ and $\mathrm{S} = |\mathrm{A}|^{1-\gamma}$ in (2.4) for $\delta\epsilon[0,1]$ and $\varepsilon \ge 1$,

$$ber^2\left(\Delta_\gamma(A)\right)$$
$$\le \frac{1}{2}\|(1-\delta)|U^*|\mathrm{A}|^\gamma|^{2\rho\varepsilon}$$
$$+ \delta|\mathrm{A}|^{2(1-\gamma)\sigma\varepsilon}\|_{ber}^{\frac{1}{\varepsilon}}\||\mathrm{A}|^\gamma\mathrm{U}\|_{ber}^{2\delta}\|A\|_{ber}^{2(1-\gamma)(1-\delta)}$$
$$\le \tfrac{1}{2}\|(1-\delta)|U^*|\mathrm{A}|^\gamma|^{2\rho\varepsilon} +$$
$$\delta|\mathrm{A}|^{2(1-\gamma)\sigma\varepsilon}\|_{ber}^{1/\varepsilon}\|\mathrm{A}\|_{ber}^{2\gamma\delta}\|A\|_{ber}^{2(1-\gamma)(1-\delta)} =$$

$\frac{1}{2}\left\|(1-\delta)|U^*|A|^\gamma|^{2\rho\varepsilon}+\right.$
$\left.\delta|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber}^{1/\varepsilon}\|A\|_{ber}^{2[\gamma\delta+(1-\gamma)(1-\delta)]}$,
which proves (2.15).

For $\gamma=\frac{1}{2}$ we have obtain the following inequalities for the Alughte transform $A^{\sim}$.

Corollary 2.7. Let $A\in\mathbb{B}(\mathcal{H})$ and $\gamma\epsilon[0,1]$, then
$ber^{2\varepsilon}(A^{\sim})\leq\left\|\frac{1}{\rho}\left|U^*|A|^{1/2}\right|^{2\rho\varepsilon}+\frac{1}{\sigma}|A|^{\sigma\varepsilon}\right\|_{ber}$,
for $\varepsilon>0$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\geq 1$. Also,

$$ber^{2\varepsilon}(A^{\sim})\leq\frac{1}{2}\left[\|A\|_{Ber}^{2\varepsilon}+ber^{\varepsilon}\left(|A|^{3/2}UU^*|A^*|^{1/2}\right)\right]$$

for $\varepsilon\geq 1$. Moreover

$$ber^{2\varepsilon}(A^{\sim})\leq\frac{1}{2}\left(\left\|\frac{1}{\rho}\left|U^*|A|^{1/2}\right|^{2\rho\varepsilon}+\frac{1}{\sigma}|A|^{\sigma\varepsilon}\right\|_{ber}+ber^{\varepsilon}\left(|A|^{3/2}UU^*|A^*|^{1/2}\right)\right)$$

for $\varepsilon\geq 1$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\geq 2$. Also, for $\delta\epsilon[0,1]$ and $\varepsilon\geq 1$, then

$$ber^2(A^{\sim})\leq\left\|(1-\delta)\left|U^*|A|^{\frac{1}{2}}\right|^{2\rho\varepsilon}+\delta|A|^{\sigma\varepsilon}\right\|_{ber}^{\frac{1}{\varepsilon}}\|A\|_{ber}.$$

For $\gamma=0$ we also have:

Corollary 2.8. If $A\in\mathbb{B}(\mathcal{H})$, then we have
$ber^{2\varepsilon}(A)\leq\left\|\frac{1}{\rho}|U^*|^{2\rho\varepsilon}+\frac{1}{\sigma}|A|^{2\sigma\varepsilon}\right\|_{ber}$,
for $\varepsilon>0$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\geq 1$. Also,

$$ber^{2\varepsilon}(A)\leq\frac{1}{2}\left[\|A\|_{Ber}^{2\varepsilon}+ber^{\varepsilon}(|A|^2UU^*)\right]$$

for $\varepsilon\geq 1$. Moreover

$$ber^{2\varepsilon}(A)\leq\frac{1}{2}\left(\left\|\frac{1}{\rho}|U^*|^{2\rho\varepsilon}+\frac{1}{\sigma}|A|^{2\sigma\varepsilon}\right\|_{ber}+ber^{\varepsilon}(|A|^2UU^*)\right)$$

for $\varepsilon\geq 1$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\geq 2$. Also, for $\delta\epsilon[0,1]$ and $\varepsilon\geq 1$, then

$$ber^2(A)\leq(1-\delta)\||U^*|^{2\rho\varepsilon}+\delta|A|^{2\sigma\varepsilon}\|_{ber}^{1/\varepsilon}\|A\|_{ber}^{2(1-\delta)}.$$

The following upper bounds for the numerical of the generalized Alughte transform are also obtained by us.:

Theorem 2.9. $\mathcal{H}=\mathcal{H}(\mathfrak{X})$ be a FHS. Let $A\in\mathbb{B}(\mathcal{H})$ and $\gamma\epsilon[0,1]$, then

$$ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)\leq\left\|\frac{1}{\rho}|A|^{2\gamma\rho\varepsilon}+\frac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber},\qquad(2.16)$$

for $\varepsilon>0$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\geq 1$. If $\varepsilon\geq 1$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\geq 2$, then

$$ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)\leq\frac{1}{2}\left(\left\|\frac{1}{\rho}|A|^{2\gamma\rho\varepsilon}+\frac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber}+\|A\|_{ber}^{2\varepsilon}\right)\qquad(2.17)$$

Also, for $\delta\epsilon[0,1]$ and $\varepsilon\geq 1$, then

$$ber^2\left(\Delta_\gamma(A)\right)\leq(1-\delta)\left\||A|^{2\gamma\varepsilon}+\delta|A|^{2(1-\gamma)\varepsilon}\right\|_{ber}^{1/\varepsilon}\|A\|_{ber}^{2[\gamma\delta+(1-\delta)(1-\gamma)]}.\qquad(2.18)$$

Proof. If we take $\mathrm{T}=|\mathrm{A}|^\gamma$ and $\mathrm{S}=\mathrm{U}|\mathrm{A}|^{1-\gamma}$ and can see that $\mathrm{TS}=|\mathrm{A}|^\gamma\mathrm{U}|\mathrm{A}|^{1-\gamma}=\Delta_\gamma(A)$ then by (2.1) we have (2.16).

If $\varepsilon\geq 1$, $\rho,\sigma>1$ with $\frac{1}{\rho}+\frac{1}{\sigma}=1$ and $\rho\varepsilon,\sigma\varepsilon\geq 2$, then by the same choice in (2.3) we have

$$\begin{aligned}ber^{2\varepsilon}\left(\Delta_\gamma(A)\right)&\leq\frac{1}{2}\left\|\frac{1}{\rho}|A|^{2\gamma\rho\varepsilon}+\frac{1}{\sigma}|A|^{2(1-\gamma)\varepsilon\sigma}\right\|_{ber}+ber^{\varepsilon}\left(|A|^{2(1-\gamma)}|A|^{2\gamma}\right)\\&\leq\frac{1}{2}\left\|\frac{1}{\rho}|A|^{2\gamma\rho\varepsilon}+\frac{1}{\sigma}|A|^{2(1-\gamma)\varepsilon\sigma}\right\|_{ber}+\|A\|_{ber}^{2\varepsilon},\end{aligned}$$

which proves (2.17).
For $\delta\epsilon[0,1]$ and $\varepsilon\geq 1$, then by (2.4) we have

$$ber^2\left(\Delta_\gamma(A)\right)\leq\frac{1}{2}\left\|(1-\delta)|A|^{2\gamma\rho\varepsilon}+\delta|A|^{2(1-\gamma)\varepsilon\sigma}\right\|_{ber}^{\frac{1}{\varepsilon}}\|A\|_{ber}^{2\delta\gamma}\|A\|_{ber}^{2(1-\delta)(1-\gamma)}$$

$$= \left\| (1-\delta)|A|^{2\gamma\varepsilon} + \delta|A|^{2(1-\gamma)\varepsilon} \right\|_{ber}^{1/\varepsilon} \|A\|_{ber}^{2[\gamma\delta+(1-\delta)(1-\gamma)]},$$

which proves (2.18).

Corollary 2.10. Let $A \in \mathbb{B}(\mathcal{H})$. Then

$$ber^{2\varepsilon}(A^{\sim}) \leq \frac{1}{2}\left(\left\|\frac{1}{\rho}|A|^{\rho\varepsilon} + \frac{1}{\sigma}|A|^{\sigma\varepsilon}\right\|_{ber}\right)$$

for $\varepsilon > 0$, $\rho, \sigma > 1$ with $\frac{1}{\rho} + \frac{1}{\sigma} = 1$ and $\rho\varepsilon, \sigma\varepsilon \geq 1$. If $\varepsilon \geq 1$, $\rho, \sigma > 1$ with $\frac{1}{\rho} + \frac{1}{\sigma} = 1$ and $\rho\varepsilon, \sigma\varepsilon \geq 2$, then by (2.17) we have

$$ber^{2\varepsilon}(A^{\sim}) \leq \frac{1}{2}\left(\left\|\frac{1}{\rho}|A|^{2\gamma\rho\varepsilon} + \frac{1}{\sigma}|A|^{2(1-\gamma)\sigma\varepsilon}\right\|_{ber} + \|A\|_{ber}^{2\varepsilon}\right).$$

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