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# A modified hybrid conjugate gradient method for solving unconstrained non-linear optimization problems has been developed

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***Abstract*** – This paper presents a new hybrid conjugate gradient method, combining the Conjugate Descent (CD) and Al-Bayati & Al-Assady (BA) methods, for solving unconstrained optimization problems. We provide a convergence analysis of the proposed method and demonstrate its effectiveness through numerical examples.

*Keywords – Optimization; Algorithm; Descent Direction; Line Search; Conjugate Gradient*

## I. Introduction

Consider the unconstrained optimization problem:

$$\min\{f(x): x \in \mathbb{R}^n\} \qquad (1)$$

Where $f: \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, bounded from below. To solve this problem we use a sequence $\{x_k\}$ which is given as follows:

$$x_{k+1} = x_k + \alpha_k d_k \qquad (2)$$

Starting form an initial vector $x_0$, Where $\alpha_k > 0$ is determined by linear search. To move from $x_k$ to $x_{k+1}$ we use direction $d_k$ given as follows:

$$d_0 = -g_0;\ d_{k+1} = -g_{k+1} + \beta_k d_k.$$

In order to determine $\alpha_k$ , we usually use strong Wolfe conditions (cf [6]) given by the flowing forms:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k \qquad (3)$$

$$|g_{k+1}^T d_k| \leq -\sigma g_k^T d_k \qquad (4)$$

Where $0 < \sigma < \frac{1}{2}$.

On the other hand, many other numerical methods for unconstrained optimization are proved to be convergent under the standard Wolfe conditions (3) and

$$g_{k+1}^T d_k > \sigma g_k^T d_k \qquad (5)$$

Now, let us denote $y_k = g_{k+1} - g_k$ and $s_k = x_{k+1} - x_k$.

There are formulas of the conjugate gradient parameter $\beta_k$ (cf [8], [2], [3], [4], respectively) which are given as follow:

$$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2},$$

$$\beta_k^{LS} = \frac{g_{k+1}^T y_k}{-g_k^T d_k},$$

$$\beta_k^{BA} = \frac{\|y_k\|^2}{d_k^T y_k},$$

$$\beta_k^{CD} = \frac{\|g_{k+1}\|^2}{-d_k^T g_k},$$

The aim of this study is to find a new combination based on the previous works of [1], [2] and [3]. Note that, we based on the convex combination of Andrei [1] using CD and BA methods. In the next section, we find the parameters $\theta_k$ and also we prove that $d_k$ satisfies the descent condition. In the end of his section, we give the CDBACC algorithm to solve the optimization problem. Section 3 is devoted to the study of the convergence analysis. Finally, to illustrate our method we give some numerical examples.

## II. A CONVEX COMBINATION

Before we start, we define $\beta_k^{New}$ as follows:

$$\beta_k^{New} = \theta_k \beta_k^{CD} + (1-\theta_k)\beta_k^{BA}, \qquad (6)$$

and $\theta_k$ is a scalar parameter satisfying

$0 \le \theta_k \le 1.$

The direction $d_k^{New}$ is given by:

$$d_0^{New} = -g_0, \qquad d_{k+1}^{New} = -g_{k+1} + \beta_k^{New} d_k. \quad (7)$$

**Theorem 1.** *If the relations (6) and (7) hold, then*

$$d_{k+1}^{New} = \theta_k d_{k+1} + (1-\theta_k) d_{k+1}. \qquad (8)$$

Using the following conjugate condition

$$y_k^T d_{k+1}^{New} = 0, \qquad (9)$$

and multiplying *(8)* by $y_k^T$ and by using *(9)* we get

$$\theta_k = \frac{(-y_k^T g_{k+1})(d_k^T g_k) + \|y_k\|^2 d_k^T y_k}{\|g_{k+1}\|^2 d_k^T y_k + \|y_k\|^2 d_k^T g_k} \qquad (10)$$

We could fix the $\theta_k$ as follows:

$$\theta_k \begin{cases} 0, \quad If\ \theta_k \le 0, \\ \dfrac{(-y_k^T g_{k+1})(d_k^T g_k) + \|y_k\|^2 d_k^T y_k}{\|g_{k+1}\|^2 d_k^T y_k + \|y_k\|^2 d_k^T g_k}, If\ 0 < \theta_k < 1 \\ 1, \quad If\ \ \theta_k > 1. \end{cases}$$

**Assumption1**.

$f(x)$ is bounded from below on the level set

$$S = \{\, x \in \mathbb{R}^n : f(x) \le f(x_0)\}.$$

**Assumption2.**

The gradient $g(x)$ is Lipschitz continuous i.e there exists a constant $L > 0$ such *that* $\|g(x) - g(y)\| \le L\|x - y\|$, for all $x, y \in \mathbb{R}^n$.

These assumptions imply that there exists a positive constant $\gamma$ such that $\|g(x)\| \le \gamma$, for all $x \in \mathbb{R}^n$.

**Theorem 2.** *Assume that Assumption 1 and 2 hold, let strong Wolfe conditions hold with $\sigma < \frac{1}{2}$ and also let there exist $\mu > 0$ such that*

$$\|s_k\|^2 \le \mu \|g_{k+1}\|^2, \quad \frac{\sigma}{1-\sigma}\mu L^2 < 1.$$

*Then $d_k^{New}$ satisfies the sufficient descent condition for all $k$.*

**Algorithm** (AlgorithmCDBACC)

Step 0.

Select $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$ and $0 < \delta < \sigma < \frac{1}{2}$.

Compute $f(x_0)$, and $g_0$ Consider $d_0 = -g_0$.

$$\alpha_0 = \frac{1}{\|g_0\|}.$$

Set the initial guess.

Step 1.

If $\|g_0\| \le \varepsilon$, then STOP.

Step 2.

Compute $\alpha_k > 0$ satisfying the strong Wolfe line search conditions (3) and (4).

Calculi $x_{k+1}, f_{k+1}, g_{k+1}, y_k$.

Step 3.

If $\|g_{k+1}\|^2 d_k^T y_k + \|y_k\|^2 d_k^T g_k = 0$, then set $\theta_k = 0$, else set $\theta_k$ as in (10).

Step 4.

Compute $\beta_k^{New}$ as in (6).

Step 5.

Compute $d_{k+1}^{New} = -g_{k+1} + \beta_k^{New} d_k$.

Step 6.

If the restart criterion of Powell

$$|g_{k+1}^T g_k| \ge 0.2\|g_{k+!}\|^2$$

is satisfied, then $d_{k+1}^{New} = -g_{k+1}$, else define $d_{k+1} = d_k$

Step 7.

Compute the initial guess

$\alpha_k = \alpha_{k-1} \frac{\|d_{k-1}\|}{\|d_k\|}$.

Step 8.

Set $k = k + 1$, and continue with step 2.

## III. CONVERGENCE ANALYSIS

**Lemma 1.** [7] *Assume that Assumption 1 and 2 hold. Consider any method of the form (1), where $d_k$ is a descent direction and $\alpha_k$ satisfies the standard Wolfe conditions (3) and (4).*

*Then we have that*

$$\sum_{k \ge 0} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < +\infty$$

**Lemma 1.** [8] *Suppose that Assumption 1 and 2 holds. If $d_k$ is a descent direction and the step size $\alpha_k$ satisfies $g_{k+1}^T d_k \ge \sigma g_k^T d_k$, $\sigma < 1$, then*

$$\alpha_k \ge \frac{1-\sigma}{L} \frac{|\, d_k^T g_k|}{\|d_k\|^2}.$$

According to the lemma (8), the conditions (4) and (7), that $\alpha_k$ obtained in the HCDBA method is not equal to zero, i.e., there exists $\lambda > 0$ such that

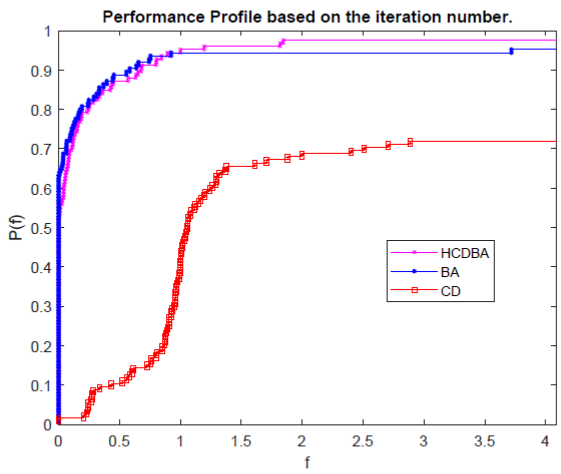
$$\alpha_k \ge \lambda, \qquad \forall k \ge 0.$$

**Theorem 1**. *Consider the iterative method of the form (1), (6), (7), (10), Let all conditions of Theorem (2) hold.*

*Then either* $g_k = 0$ *for some* $k$*, or*

$$\lim_{k\to\infty} inf\|g_k\| = 0.$$

## IV. NUMERICAL EXAMPLE




## V. CONCLUSION

The conjugate gradient methods and trust region methods are very popular now. Namely, different modifications of these methods are made, with the aim to improve them. Hybrid conjugate gradient methods are made in many different ways; this class of conjugate gradient methods is always effective. In this paper we prove the effectiveness and efficiency of the proposed hybrid conjugate gradient method.

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