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# Application of Operator Matrices of 2 × 2 of Berezin Radius Inequality

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***Abstract*** – Many researchers in mathematics and mathematical physics are working on the Berezin symbol of the core Hilbert space operator, which has been proliferating in recent years. In this direction, some researchers (1.2) continued their important studies on the Berezin inequality ([18-24]). As a matter of fact, improved and improved versions of this inequality have attracted the attention of researchers in recent years ([8-11]. In this study, the upper bounds of the Berezin radius inequalities of 2 × 2 operator matrices were found. Al-Dolat and Kittaneh ([2]) and Bani-Domi and Kittaneh ([7]) inequalities are shown for 2 × 2 operator matrices using auxiliary theorems.

*Keywords – Berezin Number, Usual Operator Norm, Arithmetic-Geometric Mean Inequality, Mixed Schwarz Inequality, Convexity*

## I. Introduction

Many researchers in mathematics and mathematical physics are working on the Berezin symbol of the core Hilbert space operator, which has been proliferating in recent years. In this direction, some researchers (1.2) continued their important studies on the Berezin inequality [18-23]. As a matter of fact, improved and improved versions of this inequality have attracted the attention of researchers in recent years ([8-11]. In this study, the upper bounds of the Berezin radius inequalities of 2×2 operator matrices were found. Al-Dolat and Kittaneh ([2]) and Bani-Domi and Kittaneh ([7]) inequalities are shown for $2 \times 2$ operator matrices using auxiliary theorems.

Recall the reproducing kernel Hilbert space (shortly, RKHS) or a functional Hilbert space (shortly, FHU ) is a Hilbert space $\mathcal{H} = \mathcal{H}(\Upsilon)$ of complex-valued function such that evalutional functional $\psi_\omega(f) = f(\omega)$, $\omega \in \Upsilon$, are continuous on a $\mathcal{H}$. By Riesz representation theorem in functional analysis, there is an unique element $k_\omega \in \mathcal{H}$ such that $f(\omega) = \langle f, \omega \rangle$, for all $f \in \mathcal{H}$. The set $\{k_\omega : \omega \in \Upsilon\}$ is called the reproducing kernel in $\mathcal{H}$. For $\omega \in \Upsilon$, $\hat{k}_\omega = \frac{k_\omega}{\|k_\omega\|}$ is denoted the normalized reproducing kernel.

**Definition 1.1**. Let $\mathcal{H}$ be a RKHS. For a bounded linear operator $V$ on $\mathcal{H}$, the Berezin symbol (or Berezin transform) is the function $\tilde{V}$ is defined by (see, [13])

$$\tilde{V}(\omega) = \langle V\hat{k}_\omega, \hat{k}_\omega \rangle_{\mathcal{H}} \ (\omega \in \Upsilon).$$

Also, the Berezin set and the Berezin number of operator V is denoted as follows, respectively.

$$Ber(V) = Range(\tilde{V}) = \{\tilde{V}(\omega) : \omega \in \Upsilon\}$$

and

$$ber(V) = sup\{|\tilde{V}(\omega)| : \omega \in \Upsilon\}.$$

For more information on Hilbert spaces with reproducing kernel and the Berezin symbol, see the work of Aronzajn [4] and Berezin [13].

For the operators on the RKHS, Karaev presented investigations on the Berezin set and Berezin number mentioned in Definition 1.1 above in [13]. For details and significant characteristics of these new concepts, see their works [16, 20, 25, 28].

The Berezin symbol $\tilde{V}$ is a bounded functional in space $\mathcal{H}$ for the arbitrarily bounded linear operator V. The characteristics of the Berezin symbol for this operator, V, symbolize some fundamental characteristics of an operator. The Berezin symbol was first introduced by F. Berezin in [13] and entered the literature as a theory with the characteristic of being a key technique in deriving several operator theory conclusions. First introduced by Karaev in [24], the Berezin set and its number are also shown as Ber(T) and ber(T), respectively. Also, Berezin number of $n \times n$ operator matrix is given by Bakherad (see, [6]).

In this study, let's assume that $(\mathcal{H}, \langle .,. \rangle)$ is a complex Hilbert space and $\mathbb{L}(\mathcal{H})$ represents the $C^*$ algebra of all bounded linear operators in $\mathcal{H}$ space. The numeric range and radius of the V operator are respectively

$$W(V) := \{\langle Vx, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

and

$$w(V) := sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}(\Upsilon) \text{ and } \|x\| = 1\}$$

is given with. The numeric range of an operator has some interesting properties, such as the closure of the numeric range of an operator's spectrum. For basic information about this theory, we can give [1,2,3,7,17] and its references.

On the other hand, for $V \in \mathbb{L}(\mathcal{H})$,

$$\frac{1}{2}\|V\| \le w(V) \le \|V\| \tag{1.1}$$

and

$$ber(V) \le w(V) \le \|V\| \tag{1.2}$$

known inequalities.

Huban et al. [22, 23] expressed the following results.

For $V, R \in \mathbb{L}(\mathcal{H})$ and $r \ge 1$,

$$\frac{1}{4}\||V|^2 + |V^*|^2\|_{ber} \le ber(V) \le \frac{1}{2}\||V|^2 + |V^*|^2\|_{ber} \tag{1.3}$$

$$ber^r(V^*R) \le \frac{1}{2}\||V|^{2r} + |R|^{2r}\|_{ber} \tag{1.4}$$

$$ber^r(V^*R) \le \frac{1}{2} ber(|R|^r + i|V|^r)^2 \le \frac{1}{2}\||V|^{2r} + |R|^{2r}\|_{ber} \tag{1.5}$$

and

$$ber(V) \le \frac{1}{2}\||V| + |V^*|\|_{ber} \le \frac{1}{2}\left(\|V\|_{ber} + \|V^2\|_{ber}^{\frac{1}{2}}\right). \tag{1.6}$$

In 2022, Basaran et al. [9] showed the following inequality.

For $V \in \mathbb{L}(\mathcal{H})$ and $r \ge 1$,

$$ber^{2r}(V) \le \frac{1}{2}\||V|^{2r} + |V^*|^{2r}\|_{ber}. \tag{1.7}$$

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ define the **2**-copies of $\mathcal{H}$**.** Based on cartesian decomposition every operator $M \in \mathbb{L}(\mathcal{H} \oplus \mathcal{H})$ has $2 \times 2$ operator matrix representation

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

with $M_{ij} \in \mathbb{L}(\mathcal{H})$, where $i, j \in \{1,2\}$. To information about the Berezin radius of operator of matrices and their application, one can refer to [6,11,15,27].

## II. Materials and Method

In this section, we will give the lemmas we need.

**Lemma 2.1** ([26])**.** Let $V \in \mathbb{L}(\mathcal{H})$ and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then

$$\langle Vx, x\rangle^r \le \langle V^r x, x\rangle \text{ for every } r \ge 1. \tag{2.1}$$

**Lemma 2.2** ([5])**.** Let $f$ be a non-negative convex function on $[0, \infty)$ and let $V, R \in \mathbb{L}(\mathcal{H})$ be a positive operators. Then

$$\left\|f\left(\frac{V+R}{2}\right)\right\| \le \left\|\frac{f(V)+f(R)}{2}\right\|. \tag{2.2}$$

**Lemma 2.3** ([14])**.** Let $a, b, c \in \mathcal{H}$ with $\|c\| = 1$. Then

$$|\langle a, c\rangle\langle c, b\rangle| \le \frac{1}{2}(\|a\|\|b\| + |\langle a, b\rangle|). \tag{2.3}$$

**Lemma 2.4** ([3])**.** Let $a, b, c \in \mathcal{H}$ with $\|c\| = 1$. Then

$$|\langle a, c\rangle\langle c, b\rangle|^r \le \frac{1}{2}\|a\|^r\|b\|^r + \frac{\gamma}{2}\|a\|^{\frac{r}{2}}\|b\|^{\frac{r}{2}}|\langle a, b\rangle|^{\frac{r}{2}} + \frac{1-\gamma}{2}|\langle a, b\rangle|^r \tag{2.4}$$

for every $r \ge 1$ and $\gamma \in [0,1]$.

**Lemma 2.5** ([3]). Let $a,b,c \in \mathcal{H}$ with $\|c\|=1$. Then

$$|\langle a,c\rangle\langle c,b\rangle|^r \leq \frac{1}{4}\|a\|^r\|b\|^r + \frac{2+\gamma}{4}\|a\|^{\frac{r}{2}}\|b\|^{\frac{r}{2}}|\langle a,b\rangle|^{\frac{r}{2}} + \frac{1-\gamma}{4}|\langle a,b\rangle|^r \quad (2.5)$$

for every $r \geq 1$ and $\gamma \in [0,1]$.

**Lemma 2.6** ([6]). Let $P \in \mathbb{L}(\mathcal{H}_1(\Upsilon))$, $R \in \mathbb{L}(\mathcal{H}_1(\Upsilon),\mathcal{H}_2(\Upsilon))$, $S \in \mathbb{L}(\mathcal{H}_2(\Upsilon),\mathcal{H}_1(\Upsilon))$ and $T \in \mathbb{L}(\mathcal{H}_2(\Upsilon))$. Then the following statements obtain:

$$ber\left(\begin{bmatrix} P & 0 \\ 0 & T\end{bmatrix}\right) = max\{ber(P), ber(T)\} \quad (2.6)$$

and

$$ber\left(\begin{bmatrix} 0 & R \\ S & 0\end{bmatrix}\right) = \frac{1}{2}(\|R\|+\|S\|) \quad (2.7)$$

**Lemma 2.7** ([11]). Let $P,R \in \mathbb{L}(\mathcal{H}(\Upsilon))$. Then

$$ber\left(\begin{bmatrix} P & R \\ R & P\end{bmatrix}\right) = max\{ber(P+R), ber(P-R)\}. \quad (2.8)$$

In particular,

$$ber\left(\begin{bmatrix} 0 & R \\ R & 0\end{bmatrix}\right) = ber(R). \quad (2.9)$$

## III. RESULTS

In the first theorem of this section, we present an upper bound for the Berezin radius of a $2\times 2$ operator matrix, which generalize [11, Theorem 2.7].

**Theorem 3.1.** Let $P,R,S,T \in \mathbb{L}(\mathcal{H}(\Upsilon))$. Then for every $r \geq 2$, we get

$$ber\left(\begin{bmatrix} P & R \\ S & T\end{bmatrix}\right) \leq 2^{r-1}max\{ber^r(P), ber^r(T)\} + 2^{r-2}max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\} + 2^{r-3}max\{\||S|^r+|R^*|^r\|_{ber}, \||R|^r+|S^*|^r\|_{ber}\}.$$

**Proof.** Let

$$M = \begin{bmatrix} P & R \\ S & T\end{bmatrix},\ M_1 = \begin{bmatrix} P & 0 \\ 0 & T\end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 0 & R \\ S & 0\end{bmatrix}.$$

For every $(\omega_1,\omega_2) \in \Upsilon_1 \times \Upsilon_2$, let $\hat{k}_\omega = \hat{k}_{(\omega_1,\omega_2)} = \begin{bmatrix} k_{\omega_1} \\ k_{\omega_2}\end{bmatrix}$ be a normalized reproducing kernel in $\mathcal{H}_1(\Upsilon) \oplus \mathcal{H}_2(\Upsilon)$. Then we have

$$|\langle M\hat{k}_\omega,\hat{k}_\omega\rangle|^r$$

$$\leq \left(\langle M_1\hat{k}_\omega,\hat{k}_\omega\rangle + \langle M_2\hat{k}_\omega,\hat{k}_\omega\rangle\right)^r$$

$$\leq 2^{r-1}|\langle M_1\hat{k}_\omega,\hat{k}_\omega\rangle|^r + 2^{r-1}|\langle M_2\hat{k}_\omega,\hat{k}_\omega\rangle|^r$$

(by inequality (2.2))

$$\leq 2^{r-1}|\langle M_1\hat{k}_\omega,\hat{k}_\omega\rangle|^r + 2^{r-1}|\langle M_2\hat{k}_\omega,\hat{k}_\omega\rangle\langle\hat{k}_\omega,M_2^*\hat{k}_\omega\rangle|^{\frac{r}{2}}$$

$$\leq 2^{r-1}|\langle M_1\hat{k}_\omega,\hat{k}_\omega\rangle|^r + 2^{r-1}\left(\frac{|\langle M_2^2\hat{k}_\omega,\hat{k}_\omega\rangle|}{2} + \frac{\|M_2\hat{k}_\omega\|\|M_2^*\hat{k}_\omega\|}{2}\right)^{\frac{r}{2}}$$

(by inequality (2.3))

$$\leq 2^{r-1}|\langle M_1\hat{k}_\omega,\hat{k}_\omega\rangle|^r + 2^{r-2}\left(|\langle M_2^2\hat{k}_\omega,\hat{k}_\omega\rangle|^{\frac{r}{2}} + \|M_2\hat{k}_\omega\|^{\frac{r}{2}}\|M_2^*\hat{k}_\omega\|^{\frac{r}{2}}\right)$$

(by inequality (2.2))

$$\leq 2^{r-1}|\langle M_1\hat{k}_\omega,\hat{k}_\omega\rangle|^r + 2^{r-2}|\langle M_2^2\hat{k}_\omega,\hat{k}_\omega\rangle|^{\frac{r}{2}} + 2^{r-3}\left(\|M_2\hat{k}_\omega\|^r + \|M_2^*\hat{k}_\omega\|^r\right)$$

(by A.M-G.M inequality)

$$\leq 2^{r-1}|\langle M_1\hat{k}_\omega,\hat{k}_\omega\rangle|^r + 2^{r-2}|\langle M_2^2\hat{k}_\omega,\hat{k}_\omega\rangle|^{\frac{r}{2}} + 2^{r-3}\langle(|M_2|^r+|M_2^*|^r)\hat{k}_\omega,\hat{k}_\omega\rangle.$$

By taking the supremum over $(\omega_1,\omega_2) \in \Upsilon_1 \times \Upsilon_2$ and than applying inequality (2.6) in the above inequality, we get

$$ber\left(\begin{bmatrix} P & R \\ S & T\end{bmatrix}\right) \leq 2^{r-1}max\{ber^r(P), ber^r(T)\} + 2^{r-2}max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\} + 2^{r-3}max\{\||S|^r+|R^*|^r\|_{ber}, \||R|^r+|S^*|^r\|_{ber}\}.$$

Theorem 3.1 can be used to determine the upper bounds for the Berezin radius inequalities of RKHS. Some of these more significant rewards can be accessed in the results below.

**Corollary 3.2.** Let $P,R \in \mathbb{L}(\mathcal{H})$. Then for every $r \geq 2$, we get

$$max\{ber^r(P+R), ber^r(P-R)\} = ber\left(\begin{bmatrix} P & R \\ R & P\end{bmatrix}\right) \leq 2^{r-1}ber^r(P) + 2^{r-2}ber^{\frac{r}{2}}(R^2) + 2^{r-3}\||R|^r+|R^*|^r\|_{ber}.$$

By putting $P=0$ and $R=V$ in the above corollary we reach the following result.

**Corollary 3.3.** Let $V \in \mathbb{L}(\mathcal{H})$. Then for every $r \geq 2$, we get

$ber^r(V) = ber\left(\begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix}\right)$
$\leq 2^{r-2} ber^{\frac{r}{2}}(V^2) + 2^{r-3} \| |V|^r + |V^*|^r \|_{ber}.$

Next, we submit new upper for the Berezin radius of the off-diagonal of a $2 \times 2$ operator matrix.

**Theorem 3.4**. Let $R, S \in \mathbb{L}(\mathcal{H})$. Then

$ber^{2r}\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right)$
$\leq \frac{1}{4} max\{ber^2(|S|^r + i|R^*|^r), ber^2(|R|^r + i|S^*|^r)\}$
$+ \frac{\gamma}{4}\ max\{\| |S|^r + |R^*|^r \|_{ber}, \| |R|^r + |S^*|^r \|_{ber}\} max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\}$
$+ \frac{1-\gamma}{2}\ max\{ber^r(RS), ber^r(SR)\},$

for every $r \geq 2$ and $\gamma \in [0,1]$.

**Proof.** Let $M = \begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}$. For every $(\omega_1, \omega_2) \in \Upsilon_1 \times \Upsilon_2$, let $\hat{k}_\omega = \hat{k}_{(\omega_1,\omega_2)} = \begin{bmatrix} k_{\omega_1} \\ k_{\omega_2} \end{bmatrix}$ be a normalized reproducing kernel in $\mathcal{H}_1(\Upsilon) \oplus \mathcal{H}_2(\Upsilon)$. Then we have

$\left|\langle M\hat{k}_\omega, \hat{k}_\omega\rangle\right|^{2r}$
$\leq \left|\langle M\hat{k}_\omega, \hat{k}_\omega\rangle\langle \hat{k}_\omega, M^*\hat{k}_\omega\rangle\right|^r$
$\leq \frac{1}{2}\|M\hat{k}_\omega\|^r \|M^*\hat{k}_\omega\|^r$
$+ \frac{\gamma}{2}\|M\hat{k}_\omega\|^{\frac{r}{2}} \|M^*\hat{k}_\omega\|^{\frac{r}{2}} |\langle M\hat{k}_\omega, M^*\hat{k}_\omega\rangle|^{\frac{r}{2}}$
$+ \frac{1-\gamma}{2}|\langle M\hat{k}_\omega, M^*\hat{k}_\omega\rangle|^r$

(by inequality (2.4))

$\leq \frac{1}{4}\left(\|M\hat{k}_\omega\|^{2r} + \|M^*\hat{k}_\omega\|^{2r}\right)$
$+ \frac{\gamma}{4}\left(\|M\hat{k}_\omega\|^r + \|M^*\hat{k}_\omega\|^r\right)|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^{\frac{r}{2}}$
$+ \frac{1-\gamma}{2}|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^r$

(by A.M-G.M inequality)

$\leq \frac{1}{4}\left(\langle |M|^r\hat{k}_\omega, \hat{k}_\omega\rangle^2 + \langle |M^*|^r\hat{k}_\omega, \hat{k}_\omega\rangle^2\right)$
$+ \frac{\gamma}{4}\langle (|M|^r + |M^*|^r)\hat{k}_\omega, \hat{k}_\omega\rangle|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^{\frac{r}{2}}$
$+ \frac{1-\gamma}{2}|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^r$

(by inequality (2.1))

$\leq \frac{1}{4}|\langle (|M|^r + i|M^*|^r)\hat{k}_\omega, \hat{k}_\omega\rangle|^2$
$+ \frac{\gamma}{4}\langle (|M|^r + |M^*|^r)\hat{k}_\omega, \hat{k}_\omega\rangle|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^{\frac{r}{2}}$
$+ \frac{1-\gamma}{2}|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^r.$

By taking the supremum over $(\omega_1, \omega_2) \in \Upsilon_1 \times \Upsilon_2$ and than applying inequality (2.6) in the above inequality, we get

$ber^{2r}\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right)$
$\leq \frac{1}{4} max\{ber^2(|S|^r + |R^*|^r), ber^2(|R|^r + |S^*|^r)\}$
$+ \frac{\gamma}{4}\ max\{\| |S|^r + |R^*|^r \|_{ber}, \| |R|^r + |S^*|^r \|_{ber}\} max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\}$
$+ \frac{1-\gamma}{2}\ max\{ber^r(RS), ber^r(SR)\}.$

Başaran and Gürdal [12] prove the following theorem.

**Theorem 3.5**. Let $\mathcal{H} = \mathcal{H}(\Upsilon)$ be an RKHS and $V \in \mathcal{L}(\mathcal{H})$. Then $\gamma \in [0,1]$ and $r \geq 1$

$$ber^{2r}(V) \leq \frac{1+\gamma}{4}\| |V|^{2r} + |V^*|^{2r} \|_{ber} + \frac{1-\gamma}{2} ber^r(V^2). \quad (3.1)$$

As special case of Theorem 3.4, we obtain the following improvement of the inequality (3.1).

**Corollary 3.6**. Let $V \in \mathbb{L}(\mathcal{H})$. Then for every $r \geq 2$ and $\gamma \in [0,1]$, we get

$ber^{2r}(V)$
$\leq \frac{1}{4} ber^2(|V|^r + i|V^*|^r)$
$+ \frac{\gamma}{4}\| |V|^{2r} + |V^*|^{2r} \|_{ber} ber^{\frac{r}{2}}(V^2)$
$+ \frac{1-\gamma}{2} ber^r(V^2)$
$\leq \frac{\gamma}{4}\| |V|^{2r} + |V^*|^{2r} \|_{ber} + \frac{1-\gamma}{2} ber^r(V^2).$

**Proof.** We have

$ber^{2r}(V) = ber^{2r}\left(\begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix}\right)$

(by inequality (2.9))

$\leq \frac{1}{4} ber^2(|V|^r + i|V^*|^r)$

$+\frac{\gamma}{4}\||V|^{2r}+|V^*|^{2r}\|_{ber} ber^{\frac{r}{2}}(V^2)$
$+\frac{1-\gamma}{2} ber^r(V^2)$
(by Theorem 3.4)
$\leq \frac{1}{4}\||V|^{2r}+|V^*|^{2r}\|_{ber}$
$+\frac{\gamma}{4}\||V|^{2r}+|V^*|^{2r}\|_{ber} ber^{\frac{r}{2}}(V^2)$
$+\frac{1-\gamma}{2} ber^r(V^2)$
(by inequality (1.5))
$\leq \frac{1}{4}\||V|^{2r}+|V^*|^{2r}\|_{ber}+\frac{\gamma}{8}\||V|^{2r}+|V^*|^{2r}\|_{ber}$
$+\frac{1-\gamma}{2} ber^r(V^2)$
(by inequality (1.4))
$\leq \frac{1+\gamma}{4}\||V|^{2r}+|V^*|^{2r}\|_{ber}+\frac{1-\gamma}{2} ber^r(V^2)$
(by inequality (2.2)).

In the following inequality, we show a new upper bound for the Berezin radius of a $2\times 2$ operator matrices.

**Corollary 3.7.** Let $P,R,S,T \in \mathbb{L}(\mathcal{H}(\Upsilon))$. Then for every $r \geq 2$ and $\gamma \in [0,1]$, we have
$ber^{2r}\left(\begin{bmatrix} P & R \\ S & T \end{bmatrix}\right)$
$\leq 2^{2r-1} max\{ber^{2r}(A), ber^{2r}(D)\}$
$+2^{2r-3} max\{ber^2(|S|^r+i|R^*|^r), ber^2(|R|^r+i|S^*|^r)\}$
$+\gamma 2^{2r-3} max\{\||S|^r+|R^*|^r\|_{ber}, \||R|^r+|S^*|^r\|_{ber}\} max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\}$
$+(1-\gamma)2^{2r-2} max\{ber^r(RS), ber^r(SR)\}$.

**Proof.** From the inequality (2.2) and Theorem 3.4, we have
$ber^{2r}\left(\begin{bmatrix} P & R \\ S & T \end{bmatrix}\right)$
$\leq \left(ber\left(\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix}\right)+ber\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right)\right)^{2r}$
$\leq 2^{2r-1} ber^{2r}\left(\begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix}\right)+2^{2r-1} ber^{2r}\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right)$
$\leq 2^{2r-1} max\{ber^{2r}(A), ber^{2r}(D)\}$
$+2^{2r-3} max\{ber^2(|S|^r+i|R^*|^r), ber^2(|R|^r+i|S^*|^r)\}$
$+\gamma 2^{2r-3} max\{\||S|^r+|R^*|^r\|_{ber}, \||R|^r+|S^*|^r\|_{ber}\} max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\}$
$+(1-\gamma)2^{2r-2} max\{ber^r(RS), ber^r(SR)\}$.

The following result offers an upper bound the Berezin radius of the sum of operators.

**Corollary 3.8.** Let $P,R \in \mathbb{L}(\mathcal{H}(\Upsilon))$. Then for every $r \geq 2$ and $\gamma \in [0,1]$, we have
$max\{ber^{2r}(P+R), ber^{2r}(P-R)\}$
$= ber\left(\begin{bmatrix} P & R \\ R & P \end{bmatrix}\right)$
$\leq 2^{r-1} ber^{2r}(P)+2^{2r-3} ber^2(|R|^r+i|R^*|^r)$
$+\gamma 2^{2r-3}\||R|^r+|R^*|^r\|_{ber} ber^{\frac{r}{2}}(R^2)$
$+(1-\gamma)2^{2r-2} ber^r(R^2)$.

Now, we can present the following result as follow.

**Theorem 3.9**. Let $R,S \in \mathbb{L}(\mathcal{H}(\Upsilon))$. Then for every $r \geq 2$ and $\gamma \in [0,1]$, we have
$ber^{2r}\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right)$
$\leq \frac{1}{8} max\{ber^2(|S|^r+i|R^*|^r), ber^2(|R|^r+i|S^*|^r)\}$
$+\frac{2+\gamma}{8} max\{\||S|^r+|R^*|^r\|_{ber}, \||R|^r+|S^*|^r\|_{ber}\} max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\}$
$+\frac{1-\gamma}{4} max\{ber^r(RS), ber^r(SR)\}$.

**Proof.** Let $M=\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}$. For every $(\omega_1,\omega_2) \in \Upsilon_1 \times \Upsilon_2$, let $\hat{k}_\omega=\hat{k}_{(\omega_1,\omega_2)}=\begin{bmatrix} k_{\omega_1} \\ k_{\omega_2} \end{bmatrix}$ be a normalized reproducing kernel in $\mathcal{H}_1(\Upsilon) \oplus \mathcal{H}_2(\Upsilon)$. Then we have

$|\langle M\hat{k}_\omega, \hat{k}_\omega\rangle|^{2r}$
$\leq |\langle M\hat{k}_\omega, \hat{k}_\omega\rangle\langle \hat{k}_\omega, M^*\hat{k}_\omega\rangle|^r$
$\leq \frac{1}{4}\|M\hat{k}_\omega\|^r\|M^*\hat{k}_\omega\|^r$
$+\frac{2+\gamma}{4}\|M\hat{k}_\omega\|^{\frac{r}{2}}\|M^*\hat{k}_\omega\|^{\frac{r}{2}}|\langle M\hat{k}_\omega, M^*\hat{k}_\omega\rangle|^{\frac{r}{2}}$
$+\frac{1-\gamma}{4}|\langle M\hat{k}_\omega, M^*\hat{k}_\omega\rangle|^r$
(by inequality (2.5))
$\leq \frac{1}{8}\left(\|M\hat{k}_\omega\|^{2r}+\|M^*\hat{k}_\omega\|^{2r}\right)$
$+\frac{2+\gamma}{8}\left(\|M\hat{k}_\omega\|^r+\|M^*\hat{k}_\omega\|^r\right)|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^{\frac{r}{2}}$
$+\frac{1-\gamma}{4}|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle|^r$

(by A.M-G.M inequality)

$\leq \frac{1}{8}\left|(|M|^r + i|M^*|^r)\hat{k}_\omega, \hat{k}_\omega\right|^2$

$+\frac{2+\gamma}{8}\langle(|M|^r + |M^*|^r)\hat{k}_\omega, \hat{k}_\omega\rangle\left|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|^{\frac{r}{2}}$

$+\frac{1-\gamma}{4}\left|\langle M^2\hat{k}_\omega, \hat{k}_\omega\rangle\right|^r$

(by inequality (2.1)).

By taking the supremum over $(\omega_1, \omega_2) \in \Upsilon_1 \times \Upsilon_2$ and than applying inequality (2.6) in the above inequality, we get

$ber^{2r}\left(\begin{bmatrix} 0 & R \\ S & 0 \end{bmatrix}\right)$

$\leq \frac{1}{8} max\{ber^2(|S|^r + i|R^*|^r), ber^2(|R|^r + i|S^*|^r)\}$

$+\frac{2+\gamma}{8} max\{\||S|^r + |R^*|^r\|_{ber}, \||R|^r + |S^*|^r\|_{ber}\}\, max\left\{ber^{\frac{r}{2}}(RS), ber^{\frac{r}{2}}(SR)\right\}$

$+\frac{1-\gamma}{4}\ max\{ber^r(RS), ber^r(SR)\}.$

**Corollary 3.10**. Let $V \in \mathbb{L}(\mathcal{H}(\Upsilon))$. Then for every $r \geq 2$ and $\gamma \in [0,1]$, we have

$ber^{2r}(V)$

$\leq \frac{1}{8} ber^2(|S|^r + i|R^*|^r)$

$+\frac{2+\gamma}{8}\||V|^r + |V|^r\|_{ber} ber^{\frac{r}{2}}(V^2) + \frac{1-\gamma}{4}\ ber^r(V^2)$

$\leq \||V|^{2r} + |V|^{2r}\|_{ber}.$

**Proof.** We have

$$ber^{2r}(V) = ber^{2r}\left(\begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix}\right)$$

(by inequality (2.9))

$\leq \frac{1}{8} ber^2(|S|^r + i|R^*|^r)$

$+\frac{2+\gamma}{8}\||V|^r + |V^*|^r\|_{ber} ber^{\frac{r}{2}}(V^2) + \frac{1-\gamma}{4}\ ber^r(V^2)$

(by Theorem 3.9)

$\leq \frac{1}{8}\||V|^{2r} + |V^*|^{2r}\|_{ber}$

$+\frac{2+\gamma}{8}\||V|^r + |V^*|^r\|_{ber} ber^{\frac{r}{2}}(V^2) + \frac{1-\gamma}{4}\ ber^r(V^2)$

(by inequality (1.5))

$\leq \frac{1}{8}\||V|^{2r} + |V^*|^{2r}\|_{ber}$

$+\frac{2+\gamma}{16}\|(|V|^r + |V^*|^r)^2\|_{ber} + \frac{1-\gamma}{4}\ ber^r(V^2)$

(by inequality (1.4))

$\leq \frac{1}{8}\||V|^{2r} + |V^*|^{2r}\|_{ber}$

$+\frac{2+\gamma}{16}\||V|^{2r} + |V^*|^{2r}\|_{ber} + \frac{1-\gamma}{8}\ \||V|^{2r} + |V^*|^{2r}\|_{ber}$

(by inequality (2.2))

$= \frac{1}{4}\||V|^{2r} + |V^*|^{2r}\|_{ber}.$

This completes the proof.

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