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A New Theorem on Third Order Differential Equation with Retarded Argument

Erdoğan Şen*

Matematik Bölümü, Tekirdağ Namık Kemal Üniversitesi, Türkiye

**esen@nku.edu.tr*

Abstract – In this work we extend the inequality which is for ordinary differential equations to differential equations with retarded argument. If the retarded argument vanishes the inequality turns to an inequality for third order ordinary differential equations.

Keywords – Differential Equation With Retarded Argument; Mean-Value Theorem; Inequality; Existence Theorem, Ordinary Differential Equation

I. INTRODUCTION

Differential equations with deviating argument, in particular differential equations with retarded argument, describe processes with after effect. Applications of differential equations with retarded argument can be encountered in the theory of automatic control, in the theory of self-oscillatory systems, in the study of problems connected with combustion in rocket engines, in a number of problems in economics, biophysics. The problems in these areas can be solved reducing differential equations with retarded argument. Equations with retarded argument appear, for example, each time when in some physical or technological problem, the force operating at the mass point depends on the velocity and the position of this point, not only at the given instant, but also at some given previous instant [5].

Third order differential equations with deviating arguments are equations of the form

$$F\left(t,x(t),\dots,x^{(m_0)}(t),x(t-\Delta_1(t)),\dots,x^{(m_1)}(t-\Delta_1(t)),\dots,x(t-\Delta_n(t)),\dots,x^{(m_n)}(t-\Delta_n(t))\right)=0$$

where $\Delta_i(t)\geq 0,\ i=1,\dots,n,$ and $\max_{0\leq i\leq n} m_i=3$. By $x^{(m_i)}(t-\Delta_i(t))$ we mean the m_ith derivative of the function $x(z)$ evaluated at the point $z=t-\Delta_i(t)$.

A natural classification of equations with deviating arguments was proposed by G.A. Kamenskii [2]. When equation is solved for $x^{(m_0)}(t)$, it becomes

$$x^{(m_0)}(t)=f\left(t,x(t),x^{(m_0-1)}(t),x(t-\Delta_1(t)),\dots,x^{(m_1)}(t-\Delta_1(t)),\dots,x(t-\Delta_n(t)),\dots,x^{(m_n)}(t-\Delta_n(t))\right)$$

Let $\eta=m_0-\mu$, where $\mu=\max_{1\leq i\leq n} m_i$. Equations for which $\eta>0$ are called equations with retarded argument, those for which $\eta=0$ are called equations of neutral type and those for which $\eta<0$ are called equations of advanced type.

Existence theorems for differential equations with retarded argument were proved by the assistance of similar inequalities which is obtained in this work [4,5]. In the case of $\Delta(t)=0$ inequalities in [6,7] turn to inequalities for second order ordinary differential equations which play important roles to prove existence theorem for second order ordinary differential equations [1,3]

In this study we consider the third order differential equation with retarded argument

$$L(y)=y'''(t)+M_1(t)y''(t-\Delta(t))+M_2(t)y'(t-\Delta(t))+M_3(t)y(t-\Delta(t))=0 \tag{1}$$

on an interval *I*. Here $M_j(t),\, \Delta(t)$ are continuous functions on I for $j = 1,2,3$ and $1 \ge \Delta(t) \ge 0$ for each $t \in I$.

II. RESEARCH FINDINGS

Theorem. *Let us denote by every point with* t_{k_i} *which is satisfying the mean-value theorem for a continuous solution* $w^{(i)}(t_k)$ *of* (1) *on* $[t_k - \Delta(t_k), t_k] \subseteq I$ *for each* $t_k \in I$, $k \in J$ *and* $i = 0,1,2$ *where* J *is an index set. Also let us assume that* $\sup_{t\in I} M_j(t) = M_{0_j}$, $j = 1,2,3$ *where* M_{0_j}s *are real numbers. Then for all* t_k *in* I

$$\|w(t_{k_i})\| e^{-\psi|t_k - t_{k_i}|} \le \|w(t_k)\| \le \|w(t_{k_i})\| e^{\psi|t_k - t_{k_i}|}$$
(2)

where

$$\|w(t_k)\| = \left[|w(t_k)|^2 + |w'(t_k)|^2 + |w''(t_k)|^2 \right]^{1/2},$$

$$\psi = 1 + \left|M_{0_1}\right| (1 + \left|w'''(t_{k_2})\right|) + \left|M_{0_2}\right| (1 + \left|w''(t_{k_1})\right|) + \left|M_{0_3}\right| (1 + \left|w'(t_{k_0})\right|).$$

Proof In the case $\Delta(t) = 0$ the theorem can be proved as being in the theory of ordinary differential equations [1]. Now let us consider the case $\Delta(t) > 0$. From the mean-value theorem we can write the followings:

$$\frac{w(t_k) - w(t_k - \Delta(t_k))}{\Delta(t_k)} = w'(t_{k_0}),$$

$$\frac{w'(t_k) - w'(t_k - \Delta(t_k))}{\Delta(t_k)} = w''(t_{k_1}),$$

$$\frac{w''(t_k) - w''(t_k - \Delta(t_k))}{\Delta(t_k)} = w'''(t_{k_2}),$$

Thus

$$w\left(t_k - \Delta\left(t_k\right)\right) = w\left(t_k\right) - w'\left(t_{k_0}\right)\Delta\left(t_k\right),$$

$$w'\left(t_k - \Delta\left(t_k\right)\right) = w'\left(t_k\right) - w''\left(t_{k_1}\right)\Delta\left(t_k\right),$$

$$w''\left(t_k - \Delta\left(t_k\right)\right) = w''\left(t_k\right) - w'''\left(t_{k_2}\right)\Delta\left(t_k\right),$$

and

$$|w(t_k - \Delta(t_k))| \le |w(t_k)| + \left|w'(t_{k_0})\right|,$$

$$|w'(t_k - \Delta(t_k))| \le |w'(t_k)| + \left|w''(t_{k_1})\right|,$$

$$|w''(t_k - \Delta(t_k))| \le |w''(t_k)| + \left|w'''(t_{k_2})\right|.$$
(3)

Now we let $u(t_k) = \|w(t_k)\|^2$. Thus

$$u = w\overline{w} + w\overline{w'} + \ldots + w''\overline{w''}.$$

where $\overline{w}(t_k) = \overline{w(t_k)}$. Then

$$u' = w'\overline{w} + w\overline{w'} + w'\overline{w'} + \ldots + w'''\overline{w''} + w''\overline{w'''},$$

From the definition of a derivative it follows that $\overline{w'} = \overline{w}'$. Also $|w(t_k)| = \left|\overline{w(t_k)}\right|$. Therefore

$$|u'(t_k)| \le 2|w(t_k)||w'(t_k)| + 2|w'(t_k)||w''(t_k)| + 2|w''(t_k)||w'''(t_k)|.$$
(4)

Since w satisfies $L(w) = 0$ we have

$$w''(t_k) = -\left[M_1(t_k) w''(t_k - \Delta(t_k)) + M_2(t_k) w'(t_k - \Delta(t_k)) + M_3(t_k) w(t_k - \Delta(t_k))\right]$$

and hence applying (3)

$$|w'''(t_k)| \le \left|M_{0_1}\right| \left|\left\{|w''(t_k)| + \left|w'''(t_{k_2})\right|\right\}\right| + \left|M_{0_2}\right| \left|\left\{|w'(t_k)| + \left|w''(t_{k_1})\right|\right\}\right| + \left|M_{0_3}\right| \left|\left\{|w(t_k)| + \left|w'(t_{k_0})\right|\right\}\right|.$$
(5)

Using (5) in the equation (4) we obtain

$$|u'(t_k)| \le 2|w(t_k)||w'(t_k)| + 2|w'(t_k)||w''(t_k)| + 2|M_{0_1}||w''(t_k)|^2 + 2|M_{0_2}||w'(t_k)||w''(t_k)| + 2|M_{0_3}||w(t_k)||w''(t_k)|$$
$$+2|M_{0_1}||w'''(t_{k_2})||w''(t_k)| + 2|M_{0_2}||w''(t_{k_1})||w''(t_k)| + 2|M_{0_3}||w'(t_{k_0})||w''(t_k)|.$$

Now applying the elementary fact that if f and g are any two functions

$$2|f||g| \le |f|^2 + |g|^2,$$

we get

$$|u'(t_k)| \le \left(1+|M_{0_3}|\right)|w(t_k)|^2 + \left(2+|M_{0_2}|\right)|w'(t_k)|^2 +$$
$$\left(1+2|M_{0_1}|+|M_{0_2}|+|M_{0_3}|+2\left[|M_{0_1}||w'''(t_{k_2})|+|M_{0_2}||w''(t_{k_1})|+|M_{0_3}||w'(t_{k_0})|\right]\right)|w''(t_k)|^2$$

or

$$|u'(t_k)| \le 2\psi u(t_k)$$

This is equivalent to

$$-2\psi u(t_k) \le u'(t_k) \le 2\psi u(t_k).$$

(6)

And these inequalities lead directly to (2). Indeed consider the right inequality which can be written as $u' - 2\psi u \le 0$. It is equivalent to

$$e^{-2\psi t}(u' - 2\psi u) = (e^{-2\psi t}u)' \le 0.$$

If $t_k > t_{k_i}$ we integrate from t_k to t_{k_i} obtaining $e^{-2\psi t_k}u(t_k) - e^{-2\psi t_{k_i}}u(t_{k_i}) \le 0$ or

$$u(t_k) \le u(t_{k_i})e^{2\psi(t_k - t_{k_i})}.$$

(7)

If we take square root of each side in (7), we get

$$\|w(t_k)\| \le \|w(t_{k_i})\| e^{\psi(t_k - t_{k_i})}, \quad t_k > t_{k_i}$$

The left inequality in (6) similarly implies

$$\|w(t_{k_i})\| e^{-\psi(t_k - t_{k_i})} \le \|w(t_k)\|, \quad t_k > t_{k_i}$$

And therefore

$$\|w(t_{k_i})\| e^{-\psi|t_k - t_{k_i}|} \le \|w(t_k)\| \le \|w(t_{k_i})\| e^{\psi|t_k - t_{k_i}|}, \quad t_k > t_{k_i}$$

which is just (2) for $t_k > t_{k_i}$. The case $t_k < t_{k_i}$ may be considered analogically.

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