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Ideal convergence and Ideal Dunford integration on Banach space.

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Abstract – In this paper, we propose on type of Dunford integration in the concept of ideal convergence This wants to construct a new convergence of functions in Banach space to definite the measurable functions. The main result is construction on the type of Dunford as the Ideal integral. Ideal Dunford integral is an application of the convergence ideal in integration but weak integration. For this been followed the usual route by first introducing the ideal Dunford integral and demonstrating for the ideal Dunford integral the most important statements related to it in the classical case. In this paper, we prove if the function f is Dunford integrable then it is ideal Dunford integrable, but conversely, this is not true. This gives the meaning of the extension of Dunford integration in our article. We are motivated by this by one important example published by Schvabik and Guoju, [20].

Keywords – Dunford Type İdeal İntegrals, I-Convergence, I-Measurable Function,I-Cauchy Convergence

I. INTRODUCTION

This paper was inspired by [7] and [5] where the concepts I-convergence of the sequences of real numbers and I-convergence of the function of real values. We will often quote some results from [5] that can be transferred to function in Banah space. In [7] it is shown that our I-convergence is, in a sense, equivalent to the μ-statistical convergence of J. Connor ([15]). The concept of statistical convergence is introduced in [9] and [13] and developed [17]. The concept of I-convergence is a generalization of statistical convergence and it is based on the notion of the ideal I of subsets of the set N of a positive integer.

II. PRELIMINIARES

Definition 1.

(a) Let Y be a set that is not the empty set, Y≠∅. Family$\Im\subset\Pi$(Y) is called *the ideal of the set* Y if and only if that for A, B∈$\Im$ it follows that, A∪B∈$\Im$ and for every A∈$\Im$and B ⊂ A we will have B ∈$\Im$.

(b) The ideal $\Im$ is called *non-trivia ideal* if and only if,$\Im$≠∅and y∉$\Im$. A non-trivial ideal is called *acceptable* when it contains sets with only one point on it.

Let them (T,Σ,μ) be a space with probabilistic measure μ, where T is a random set on a line, Σ-Borel'salgebra and μ is a defined measure.

I-Convergence of Sequences of Elements in Banach Space.

Definition 2. A sequence $x=(x_n)$,n∈N of elements of X is said to be I-convergent to L∈ X if and only if for each $\varepsilon>0$ the set $A(\varepsilon)=\{n\epsilon N:\|x_n-L\|\geq\varepsilon\}$ belongs to I. The element L is called the I-limit of the sequence x = {xn},n∈N. I-lim x_n =L.

Definition 3. A sequence $x=(x_n)$,n∈N of elements of X is said to be I-Caushy if for each $\varepsilon>0$, there exists q ∈ N such that $\{n\epsilon N:\|x_n-x_q\|\geq\varepsilon\}\in I$

Definition 4. A sequence $x=(x_n)$,n∈N is called weakly I-covergent if the sequence $x^*(x_n)$ is I-convergent for every $x^*\in X^*$.

Now, we deals with the generalization of the Ideal convergence of functions on normed space.
The sequence of functions {fk} contains the functions with value in vectorial space.

Definition 4: The function $f: T \to X$, where X is a vector space is called *a simple function* according to μ, if for every family of measurable sets $\{E_i\}$that has no common point ,$E_i \subset T$ and $E_i \cap E_j = \emptyset$, for $i \neq j$,where$T = \bigcup_{i=1}^{n} E_i$and $f(t) = x_i$, for $t \in E_i$,i=1, 2,…, n.

As we know before, the simple function is defined $f(t) = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}$is a characteristic function of E_i.

Definition 5: The function $f: T \to X$ is called $\mathfrak{I}$-*measurable on* T , if for every t∈T, $\varepsilon > 0$ and $A \subset \mathfrak{I}$ there is a sequence of simple functions $f_n: T \to X$ for which we have

$\|f_n(t) - f(t)\| < \varepsilon$ for $n \in \mathbb{N} \backslash A$.

Definition 6. The subsequence $\left(f_{n_k}\right)_{k \in \mathbb{N}}$of the sequence$(f_n)_{n \in \mathbb{N}} \xrightarrow{\mathfrak{I}} f$ is called fundamental if,for$A' = \{n_1 < n_2 < \cdots < n_k < \cdots\}$; $f_{n_k} \xrightarrow{\mathfrak{I}} f$ for $n \in \mathbb{N} \backslash A'$ where$A' \subset A$.

Definition 7..Let(I, Σ, μ)be a measurable complete space with a non-negative measure**.** The sequence of measured functions $(f_n)_n$in I is ***$\mathfrak{I}$-convergent according to the measure μ*** to the function f, if for each ε> 0 and σ> 0 there is an essential subsequence$\left(f_{n_k}\right)_k$of the sequence$(f_n)_n$ such that: $\mu\left\{t: \left\|f_{n_k}(t) - f(t)\right\| \geq \sigma\right\} < \varepsilon$ for $n_k \in \mathbb{N} \backslash A'$ and t ∈I. We denote $f_n(t) \xrightarrow{\mathfrak{I}-\mu} f(t)$.

Definition 8. The sequence of measured functions$(f_n)_n$with values in $Banah\ space$ is called $\mathfrak{I}$-*fundamental according to the measure* μ , $S \subset \mathfrak{I}$,if there is a natural number $(\sigma, S) \subset \mathbb{N} \backslash A$ and there is a subsequence $\left(f_{n_k}\right)_k$of $(f_n)_n, if\ \forall \varepsilon > 0$ and $\sigma > 0$, $\mu\left\{t: \left\|f_{n_k}(t) - f(t)\right\| \geq \sigma\right\} < \varepsilon$.

3.Ideal Dunford integration in Banach space.

Lemma 1. (Dunford) [12] Assume that $f: T \to X$ is I-weakly measurable where X normed space and for each $x^* \in X^*$ the function $x^*(f): T \to R$ is ideal (Bohner) integrable ($x^*(f) \in L_1$), then for each measurable E ⊂ T there exists a unique $x^{**} \in X^{**}$ such that

$x_E{}^{**} = IB \quad - \int_E \ x^*(f)$ për çdo $x^* \in X^*$.
(1)

Proof . For a given measurable E ⊂ T we have $\int_E \ x^*(f) = \int_E \ x^*(f \ \cdot \chi_E)$ and we can define

$S_E(x^*) \to x^*(f \ \cdot \chi_E)$,

where S_E is a linear map X^* into the space L_1 Lebeg(Bochner) integrable on T and

$S_E: x^* \to \int_T \ x^*(f \cdot \chi_E)\ )$

is a linear function on X^* . Assume that $x^*{}_{n_k} \to x^*$ in X^* and $S_E(x^*{}_{n_k}) \to g$ in L_1 where $n_k \in N \setminus A', k \to \infty$ also $\left(x^*{}_{n_k}\right)$ and $\left(S_E(x^*{}_{n_k}\right)$ are the essential subsequences of the sequences $\left(x^*{}_n\right)$and $\left(S_E(x^*{}_n\right)$, thus $\lim_K \int_T \left|x^*{}_{n_k}\left(f \cdot \chi_E\right) - g\right| = 0$ $n_k \in N \setminus A'$ and k→ ∞. Then $x^*{}_{n_k}\left(f \cdot \chi_E\right)$converges for $n_k \in N \setminus A'$and k→ +∞ in measure to g and by the Riesz theorem there is a subsequence esential $\left\{x^*{}_{m_r}\right\}$of $\left\{x^*{}_{n_k}\right\}$ such that

$x^*{}_{m_r}\left(f(t) \cdot \chi_E(t)\right) \to g(t)\ m_r \in N \setminus A'$ dhe

r→ +∞ for almost all t∈ T. Since $x^*{}_n \left(f(t) \cdot \chi_E(t)\right) \to g(t) \to x^*(f(t) \cdot \chi_E(t))$ for all t ∈ T, is follows $g(t) = X^*(f(t) \cdot \chi_E(t))$, for allmost t∈ T and $x^*(f \cdot \chi_E) \in L_1$. This means that the graf of the linear map $S_E: X^* \to L_1$ is closed and by the Banach closed graph theorem the operator S_E is bounded.. Hence $|S_E(x^*)| = \left|\int_T \ x^*(f \cdot \chi_E)\right| \leq \int_T \left|x^*(f \cdot \chi_E)\right| = \int_T \ |S_E(x^*)| = \|S_E(x^*)\|_{L_1} \leq \|S_E\| \|X^*\|$ and it follows $\left|\int_E \ x^*(t)\right| \leq \|S_E\| \|x^*\|$ Therefore operator

$\int_E \ x^*(t)$ is a continuous linear functional on X^* defining an element $x^{**} \in X^{**}$ for wich (1) holds. The previous Dunford lemma 1.makes it possible to introduce the following definition.

Definition . If f :T→ X is weakly ideal measurable and such that the function $x^*(f): T \to R$ is ideal Bochner integrable for each $x^* \in X^*$ then f is called ideal *Dunford integrable.*

The ideal Dunford integral DI- $\int_E f$ of f over a measurable set E⊂ T is defined by the element $x^{**} \in X^{**}$ given in Lemma (1) $x^{**} = DI - \int_E f$,where $x^{**}{}_E(x^*) = BI - \int_E x^*(f)$ për $x^* \in X^*$. For f ∈ DI we have $x^*(f) \in L_1$ for all $x^* \in X^*$. Let us define $S(x^*) = x^*(f)$, $x^* \in X^*$ (2)

Where $S: X^* \to L_1$ is a linear operator which is bounded according to the Banach closed graph theorem.Let $S^*: L^*{}_1 = L_\infty \to X^{**}$ be the adjoint of the operator S defined by

$$S^*(g)(x^*) = BI - \int_S g \cdot S(x^*)$$
$$= BI - \int_S g \cdot x^* f \in R \quad , g \in L^*{}_1$$
$$= L_\infty$$

$S^*(g)$ is a linear functional on X^* for any g ∈ $L_\infty(L^*{}_1)$ because

$$BI - \int_S g(ax^*{}_1 + bx^*{}_2)(f)$$
$$= a\int_S gx^*{}_1(f) + b\int_S g\, x^*{}_2(f)$$

and it is also bounded because the boundedness of the operator S gives

$$|S^*(g)(x^*)| = \left| BI - \int_S gT(x^*) \right|$$
$$\le \|g\|_{L_\infty} \cdot \|S(x^*)\|_{L_1}$$
$$\le \|g\|_{L_\infty} \|S\| \|x^*\|_{X^*}$$

Hence $S^*(g) \in X^{**}$ for every g ∈ L_∞ .

Assuming $g = \chi_E \in L_\infty$, where E ⊂ T is measurable, we have

$$S * (\chi_E)(x *) = BI - \int_S \chi_E x * (f) = \int_E x * (f)$$

Then $S * (\chi_E) \in X **$ for every measurable $E \subset T$ and $\nu(E) = S * (\chi_E) = DI - \int_E f$

The function $\nu(E) = DI - \int_E f$ defined for all measurable E ⊂ T , is called the *indefinite Ideal Dunford integral* of f .

The ideal Dunford integral is not countably additive in general.
The following example modified from a classic example.[schvabik].

Example. Using the function

$$f(t) = \begin{cases} n^2 \chi_{]0,\frac{1}{n}[}(t) & n\ prim \\ n\chi_{]0,\frac{1}{n}[}(t) & n\ others \end{cases}$$

f(o)=(0)

Is easy to see that

$$\int_0^{\frac{1}{k}} x^*(f(t))dt =$$

$$\begin{cases} \int_0^{\frac{1}{k}} \sum_{n\in P} \frac{1}{n^3} n^2 \chi_{\left]0,\frac{1}{n}\right]}(t)dt \\ \int_0^{\frac{1}{k}} \sum_{n\in \mathbb{N} \setminus P} \frac{1}{n^3} n\ \chi_{\left]0,\frac{1}{n}\right]}(t)dt \end{cases} = \begin{cases} \sum_{n=1}^{k-1} \frac{1}{n^3} n^2 \frac{1}{k} + \sum_{n=k}^{\infty} \frac{1}{n^3} \\ \sum_{n=1}^{k-1} \frac{1}{n^3} n \frac{1}{k} + \sum_{n=k}^{\infty} \frac{1}{n^3} \end{cases}$$

We see that

$$DI - \int_0^{\frac{1}{k}} f(t)dt = (\frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \ldots, \frac{\theta_k}{k}, 1.1\ldots) \in l_\infty$$

Where

$$\theta_k = \begin{cases} \frac{(k-1)^2}{k} & \text{për}\ \ k\ \ prim \\ \frac{k-1}{k} & \text{për}\ \ \text{të tjerë} \end{cases}$$

Proposition. Assume that $f: T \to X$ is Ideal Dunford integrable.Then the following assertions are equivalent.

a) The operator $T: X^* \to L_1$ given in formula (3) is ideal weakly compact .

b) The adjoint operator $T^*: L_\infty \to X^{**}$ is ideal weakly compact;

c) The set $\{x*(f) : x* \in B(X*)\} \subset L_1$ është uniformly integrable, i.e

$$\lim_{\mu(E)\to 0} \int_E x^*(f) = 0,$$

Is uniformly for every $x^* \in B(X^*)$

d) The indefinite Ideal Dunford integral ν(E) given by (3) is countably additive, i.e if $E_n \subset T, n \in \mathbb{N}$ are pairwise disjoint measurable sets then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n),$$

in X** (the series $\sum_{n=1}^{\infty} \upsilon(E_n)$ is converges on the norm in X**).

Proof.

Lemma.1 The set K ⊂ L_1 is weakly compact if and only if K is bounded and the integral is countably additive $\int_E f d\mu$ is uniformly for every f ∈ K, i.e for any sequences of sets are pairwise disjoint $E_n \subset S$ measurable sets ,we have

$$\lim_{n\to\infty} \int_{E_n} f d\mu = 0$$

Is uniformly for every f ∈ K

Proof.: Let us mention that by Gantmacher's theorem ([DS] .VI .Theorem 4.8.) an operator T is weakly compact if and only if its adjoint T* is weakly compact and therefore a) and b) are equivalent.

Let us consider the set

$$T(B(X*)) = \{x*(f); x* \in B(X*)\} \subset L_1$$

We have

$$\|x*(f)\|_{L_1} = \int_S |x*(f)| = \|T(x*)\|_{L_1} \le \|T\| < +\infty$$

for x*∈B(X*) because the operator T is bounded. Hence the set $T(B(X*))$ is bounded.

By the lemma 1 the set K ⊂ L_1 is weakly compact if and only if K is bounded and the integral is countably additive $\int_E f d\mu$ is uniformly for $f \in K$, i.e for any sequences of sets are pairwise disjoint $E_n \subset S$ measurable sets ,we have

$$\|x*(f)\|_{L_1} = \int_S |x*(f)| = \|T(x*)\|_{L_1} \le \|T\| < +\infty$$

for x*∈B(X*) because the operator T is bounded. Hence the set $T(B(X*))$ is bounded.

By the lemma 3.4. the set K ⊂ L_1 is weakly compact if and only if K is bounded and the integral is countably additive $\int_E f d\mu$ is uniformly for $f \in K$, i.e for any sequences of sets are pairwise disjoint $E_n \subset S$ measurable sets ,we have

$$\lim_{n\to\infty} \int_{E_n} f d\mu = 0$$

Uniformly for $f \in K$.

The set $T(B(X*)) \subset L_1$is weakly compact (is equivalent to ideal weakly compact according to the sequences) if and only if we have

$$\lim_{\mu(E)\to 0} \int_E x^*(f)$$

uniformly for every $x* \in B(X*)$.

This means that c) is equivalent to a). Assume that c)holds . It is easy to see that

$\lim_{\mu(E)\to 0} \int_E |x*(f)| = 0$ uniformly for $x* \in B(X*)$.then for every $\eta > 0$ there is an $\varepsilon > 0$ such that

$$\left|S*(\chi_E(x*))\right| = \left|\int_E x*(f)\right| < \eta$$

For $x* \in B(X*)$ .If $\mu(E) < \varepsilon$ then $\|S*(\chi_E)\| < \eta$. If $E_n \subset S, n \in \mathbb{N}$ are pairwise disjoint measurable sets denote $E = \bigcup_{n=1}^{\infty} E_n$,then

$$\lim_{n\to\infty} \mu(E \setminus \bigcup_{n=1}^{N} E_n) = 0$$

Consequently

$$\lim_{n\to\infty}\|\upsilon(E\setminus \bigcup_{n=1}^{\infty} E_n)\|_{X^{**}} = 0.$$

Since

$$E = (E\setminus \bigcup_{n=1}^{N} E_n) \cup \bigcup_{n=1}^{\infty} E_n,$$

we have by the operator finite additivity

$$S*(\chi_E) = S*\left(\chi_{E\setminus \bigcup_{n=1}^{N} E_n}\right) + S*\left(\chi_{\bigcup_{n=1}^{N} E_n}\right)$$
$$= S*\left(\chi_{E\setminus \bigcup_{n=1}^{N} E_n}\right) + \sum_{n=1}^{N} S*\left(\chi_{E_n}\right).$$

This means that

$$\nu(E) - \sum_{n=1}^{N}\nu(E_n) = \nu\left(E\setminus\bigcup_{n=1}^{\infty}E_n\right)$$

And

$$\lim_{N\to+\infty}\left\|\nu(E) - \sum_{n=1}^{N}\nu(E_n)\right\| = \lim_{N\to+\infty}\left\|\nu\left(E\setminus\bigcup_{n=1}^{\infty}E_n\right)\right\| = 0$$

This means $\nu(E)$ is countably additive .

Assume now that c) does not hold. Then there is a $k > 0$ and a sequence $E_n \subset S, n \in \mathbb{N}$ of meausurable sets with $\mu(E_n) \to 0, n \to \infty$, $\int_{E_n}|x_n^*(f)| > k$ for some $x_n^* \in B(X*)$.

Since of measures E_n tend to zero, it is possible to take a subsequence of E_m assuming that for $m< n$ we have $\int_{E_m}|x_n^*(f)| < \frac{k}{2^{n+1}}$.

Take $A_n = E_n\setminus \bigcup_{m<n} E_m$, $A_n \subset S$ measurable and $A_n \cap A_r = \emptyset, m \neq r$ and

$$\int_{A_n}|x_n^*(f)| = \int_{E_n}|x_n^*(f)| - \int_{\bigcup_{m<n} E_m}|x_n^*(f)| > \frac{k}{2}$$

Hence there exist $B_n \subset A_n, B_n$ B_n, are pairwise disjoint) such that $\int_{B_n}|x_n^*(f)| > \frac{k}{4}$

And $\left\|S*\left(\chi_{B_n}\right)\right\| > \frac{k}{4}$ for every n. Therefore the series

$$\sum_{n=1}^{\infty} S^*(B_n) = \sum_{n=1}^{\infty} \upsilon(B_n)$$

Cannot converge and d) is not satisfied. This gives the equivalence of c) and d).

III. CONCLUSION

First, we find another applicatation of Ideal Dunford integral in Banach space. We proved the same results of classic Dunford integral on Banach space for the Ideal Bohner integrable wich are more general.

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