All Sciences Proceedings
All Sciences Proceedings
http://as-proceeding.com/

1st International Conference on Scientific and Innovative Studies
April 18-20, 2023 : Konya, Turkey

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ON THE IDEAL CONVERGENCE OF MARTINGALES

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Abstract – This paper deals with the ideal convergence theorem of martingales of ideal Bochner integrable functions.First, a characterization of the properties of ideal convergent martingales is obtained.In this paper martingales of ideal Bochner integrable functions with values in a Banach space are treated.

Keywords –Martingales , Ideal Bochner Integral, Ideal Convergence ,Ideal Cauchy Convergence

I. INTRODUCTION

This paper was inspired by [7] and [5] where the concepts I-convergence of the sequences of real numbers and I-convergence of the function of real values. In [7] it is shown that our I-convergence is, in a sense, equivalent to the μ-statistical convergence of J. Connor ([15]). The concept of statistical convergence is introduced in [9] and [13] and developed [17]. The concept of I-convergence is a generalization of statistical convergence and it is based on the notion of the ideal I of subsets of the set N of positive integers. Recently martingale theory is having an important impact on Banach space theory. We will present in this paper some aspects of the basic theory of martingales of ideal Bochner integrable functions concerningideal convergence. In the fourth section martingales of statistical Bochner integrable functions are considered.

II . Definitions and Notations

Definition 1.

(a) Let Y be a set that is not the empty set, Y≠∅. Family$\mathfrak{I}\subset\Pi$(Y) is called *the ideal of the set* Y if and only if that for A, B$\in\mathfrak{I}$ it follows that, A∪B$\in\mathfrak{I}$ and for every A$\in\mathfrak{I}$and B ⊂ A we will have B $\in\mathfrak{I}$.

(b) The ideal $\mathfrak{I}$ is called *non-trivia ideal* if and only if,$\mathfrak{I}$≠∅and y$\notin\mathfrak{I}$. A non-trivial ideal is called *acceptable* when it contains sets with only one point on it.

Let them(T,Σ,μ)be a space with probabilistic measure μ, where T is a random set on a line, Σ-Borel'salgebra and μ is a defined measure.

Definition 2.A sequence $x=(x_n)$,n∈N of elements of X is said to be I-convergent to L∈ X if and only if for each $\varepsilon>0$the set $A(\varepsilon)=\{n\epsilon N:\|x_n-L\|\geq\varepsilon\}$belongs to I. The element L is called the I-limit of the sequence x = {xn},n∈N. I-lim x_n =L.

Definition 3. A sequence $x=(x_n)$,n∈N of elements of X is said to be I-Caushy if for each >0 , there exists q $\in N$ such that $\{n\epsilon N:\|x_n-x_q\|\geq\varepsilon\}\in I$

Now, we deal with the generalization of the Ideal convergence of functions on normed space.
The sequence of functions {fk} contains the functions with value in vectorial
space.

Definition 4. The subsequence $\left(f_{n_k}\right)_{k\in\mathbb{N}}$of the sequence$(f_n)_{n\in\mathbb{N}}\overset{\mathfrak{I}}{\to}f$ is called fundamental if,for $A'=\{n_1<n_2<\cdots<n_k<\cdots\};f_{n_k}\overset{\mathfrak{I}}{\to}f$ for $n\in\mathbb{N}\backslash A'$ where$A'\subset A$.

Definition 5..Let(I,Σ,μ)be a measurable complete space with a non-negative measure. The sequence of measured functions $(f_n)_n$ in I is ***𝕴-convergent according to the measure*** $\boldsymbol{\mu}$ to the function f, if for each ε> 0 and σ> 0 there is an essential subsequence $(f_{n_k})_k$ of the sequence $(f_n)_n$ such that: $\mu\{t: \|f_{n_k}(t) - f(t)\| \geq \sigma\} < \varepsilon$ for $n_k \in \mathbb{N}\backslash A'$ and t ∈I. We denote $f_n(t) \xrightarrow{\Im-\mu} f(t)$.

Definition 6. The sequence of measured functions$(f_n)_n$with values in *Banah space* is called $\Im$-*fundamental according to the measure* $\mu, S \subset \Im$,if there is a natural number $(\sigma, S) \subset \mathbb{N}\backslash A$ and there is a subsequence $(f_{n_k})_k$of$(f_n)_n, if\ \forall\varepsilon > 0$ and $\sigma > 0$, $\mu\{t: \|f_{n_k}(t) - f(t)\| \geq \sigma\} < \varepsilon$.

Lemma Salat:A sequence $x = (x_k)$ is ideal convergent to L if and only if there exists a set $\{k_1 < k_2 < \cdots < k_n < \cdots\}$;

Let X a Banach space and (Ω ,∑ ,μ) is a finite measure space and let $\{B_n, n \in N\}$be a filtration of Σ and let ($f_n; n \geq 0$) be a sequence of variables.

Definition7: A process $(f_n, B_n, n \in N)$ is called a martingale if

1. $(f_n; n \geq 0)$ is adapted to the filtration $\{B_n, n \in N\}$
2. $E\ (|f_n| < \infty)$, ∀ n
3. ∀ n ≤ n+1 ∀ A ϵB_n , $\int_A f_n d\mu = \int_A f_{n+1}\, d\mu$

Definition 8 .A martingale $(f_n, B_n, n \in N)$ is convergent if there exist $f \in L_p(\mu, X)$ suth that $\|f_n - f\| < \varepsilon$.

Properties of ideal convergent martingales.

In this section we consider martingales of ideal Bochner integrable functions.

Let be a Banach space and is a finite measure space (Ω ,∑,μ).A μ -measurable function is called ideal Bochner integrable if there exists a sequence $f_n$of simple functions such that

$$I - \lim_n \int_\Omega \|f_n - f\| d\mu$$

In this case $I - \int_E f\, d\mu$ is defined for each $E \in \sum$by

I-$\lim_K \int_\Omega \|f_n - f\| d\mu = 0$

The set of ideal Bochner integrable function is a linear space we denote with$L_p{}'(\mu, X)$ .It is clear that $I - \lim_K((\int_E |f(x)|^p)^{1/p} d\mu$ is also norm (a semi-norm) in this space. We denote it by $\|f\|'_p$.

We prove that $L_p(\mu, X) \subset L_{p'}(\mu, X)$in the same way as is shown in [2] for L_1 space.

Let (X ,‖. ‖) be a Banach space and (Ω , ∑ , μ) is a finite measure space and $\{B_n, n \in N\}$ is a monotone increasing net of sub-σ-fields of ∑ .

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Definition. A martingale $(f_n, B_n, n \in N)$ is ideal convergent in $L_p{}'(\mu, X)$- norm if there exist $f \in L_p{}'(\mu, X)$ such that $\{k\, ;\ \|f_k(t) - f(t) \geq \varepsilon\|_p{}'\} \epsilon\, I$

Since $(f_n, B_n, n \in N)$ is ideal convergent martingale in in $L_p{}'(\mu, X)$,$(1 \leq p < \propto)$ we have that if $E \in \bigcup_n B_n$ then the limit $I - \lim_n \int_E f_n\, d\mu =$ F(E) exists.

By the above definition,we have that if a martingale is convergent then it is ideal convergent,but the conversely is not true.

Proposition 1 : If $(f_n, B_n, n \in N)$ and $(g_n, B_n, n \in N)$ are ideal convergent martingales in $L_p{}'(\mu, X)$ then the sum $(f_n + g_n, B_n, n \in N)$ is idealconvergent martingale.

Proof. Since $(f_n, B_n, n \in N)$ and $(g_n, B_n, n \in N)$ are ideal convergent martingales in $L_p{}'(\mu, X)$ there exists $f, g \in L'_1$ such that , for every $\varepsilon > 0$, $\{k : \|f_k(x) - f(x)\|'_1 \geq \varepsilon\} \epsilon I$ and $\{k : \|g_k(x) - g(x)\|'_1 \geq \varepsilon\} \epsilon I$.

First we show that $(f_n + g_n, B_n, n \in N)$ is a martingale . By the linearity of conditionalexpectation we have $for\ every\ n \geq m$ $E(f_n \backslash B_m) + E(g_n \backslash B_m) = f_m + g_m$

So $E(f_n + g_n \backslash B_m) = f_m + g_m$.

For proving that

$$\{k : \|(f_k(x) + g_k(x)) - (f(x) + g(x)\|_1{}' \geq \varepsilon\} \in I$$

$\|(f_k(x) + g_k(x)) - (f(x) + g(x)\|_1{}' = \|(f_k(x) - f(x)) + (g_k(x) - g(x)\|_1{}' \leq \|f_k(x) - f(x)\|'_1 + \|g_k(x) - g(x)\|'_1 < \varepsilon/2 + \varepsilon/2 = \varepsilon$, for every $\in N\backslash A$, $A \in I$.

Let B be a sub-σ-fields of $\sum$ and $f \in L_p{}'(\mu, X)$,$(1 \leq p < \propto)$. An element $g \in L_p{}'(\mu, X)$ is called the I-conditional expectation of f relative to B if g is I-measurable and

$$I - \int_E g\, d\mu = \text{I-}\int_E f\, d\mu \text{ for all } E \in B$$

g is denoted by $E(f \backslash B)$.

Proposition 2. If $(f_n, B_n, n \in N)$ is a ideal convergent martingale in L'_1 then $(f_n - f_m, B_n, n \geq m)$ is also a ideal convergent martingale in L'_1

Proof. Since $(f_n, B_n, n \in N)$ is a ideal convergent martingale there exists $f \in L_1{}'(\mu, X)$ such that

$\|f_n - f\|'_1 < \varepsilon$ for $\in N\backslash A_1$, $A_1 \in I$. Also f_n is ideal Cauchy , so for every $\varepsilon > 0$, there exists N

such that $\|f_n - f_N\|'_1 < \varepsilon, n \in N\backslash A_2$, $A_2 \in I$.

$N\backslash A_1 \cup N\backslash A_2 \subset$ N$\backslash(A_1 \cup A_2)$, for n∈ N$\backslash(A_1 \cup A_2)$ we have

$\|f_n - f_m\|'_1 \leq \|f_n - f_N\|'_1 + \|f_N - f_m\|'_1 < \varepsilon/2 + \varepsilon/2 = \varepsilon$

We demonstrated $(f_n - f_m, B_n, n \geq m)$ is a ideal convergent martingale.

III . CONCLUSION

In this paper we are extending in case of ideal convergence the result present in [20] on Banch space.

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