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Sharper Inequalities For Berezin Radius Powers

Mehmet Gürdal[1] and Hamdullah Başaran[1,*]

[1]Department of Mathematics/Suleyman Demirel University, Turkey

[*](07hamdullahbasaran@gmail.com) Email of the corresponding author

Abstract – We provide numerous sharp inequalities that expand prior inequality using powers for the Berezin radius of functional Hilbert space operators.

Keywords – Berezin Number, Berezin Transform, Functional Hilbert Space, Mixed Schwarz Inequality, Jensen Inequality.

I. Introduction

In many areas related to operator theory, such as mathematical inequalities, functional analysis, numerical analysis, differential equations, applied mathematics, and mathematical physics, to name a few, Berezin transforms have been crucial. We begin by introducing certain notions and features of operators on a Hilbert space in order to characterize the Brezin number and norm.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{B}(\mathcal{H})$ define the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Recall the functional Hilbert space (briefly, FHS) $\mathcal{H} = \mathcal{H}(\mathrm{F})$ is a Hilbert space on some set (nonempty) F, such that evalution functional $\psi_\varsigma(f), \varsigma \in$ F, are continouns on a $\mathcal{H}$. Then by Riesz representatiom theorem , for each $\varsigma \in$ F, tehere is an unique element $k_\varsigma \in \mathcal{H}$ such that $f(\varsigma) = \langle f, k_\varsigma \rangle$, for all $f \in \mathcal{H}$. The family $\{k_\varsigma : \varsigma \in \mathrm{F}\}$ is called the reproducing kernel in $\mathcal{H}$. For $\varsigma \in$ F, $\widehat{k_\varsigma} = \frac{k_\varsigma}{\|k_\varsigma\|}$ is defined the normalized reproducing kernel.

For $V \in \mathbb{B}(\mathcal{H})$, the function $\tilde{V}$ defined on F by $\tilde{V}(\varsigma) = \langle V\widehat{k_\varsigma}, \widehat{k_\varsigma} \rangle$ is the Berezin symbol of V. Berezin symbol firstly has been introduced by Berezin ([2]). The Berezin set and Berezin number of the operator V are defined by

$$Ber(V) = \{\tilde{V}(\varsigma) : \varsigma \in \mathrm{F}\}$$

and

$$ber(V) = sup\{\tilde{V}(\varsigma) : \varsigma \in \mathrm{F}\} = sup_{\varsigma \in \mathrm{F}} \left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma} \rangle\right|$$

respectively (see, [6]). For the Toeplitz and Hankel operators on the Hardy and Bergmann spaces, the Berezin symbol has been thoroughtly researched. It is frequently used in many fields of study and uniquely identifies an operator. We recommed the reader to [3,5] for more information on the Berezin symbol.

In a FHS, the Berezin range and Berezin number of an operator V are a subset of numerical range and numerical radius of V, respectively. There are interesting propertiesof numerical range. For basic properties numerical radius, we refer to [1,4, 7]. The fact that

$$ber(V) \leq w(V) \leq \|V\|, \tag{1.1}$$

Is significant. It is well-known that for all $V \in \mathbb{B}(\mathcal{H})$,

$$ber(V) \leq \frac{1}{2}\left(\|V\| + \|V^2\|^{\frac{1}{2}}\right)$$

(see [5, Theorem 4]). Recently, Gürdal [5, Theorem 1] generalized some inequalities for powersof the Berezin radius. It has been shown that if V is any operator in $\mathbb{B}(\mathcal{H})$, and f_1, f_2 are nonnegative continouns functions on $[0,\infty)$ fulfiling $f(t)g(t) = t, (t \geq 0)$, then we get

$$ber^{\rho}(V) \leq \frac{1}{2}\left\|{f_1}^{2\rho}(|V|) + {f_2}^{2\rho}(|V^*|)\right\|_{ber}, \quad (1.2)$$

for $\rho \geq 1$. Moreover, it has been demonstrated in [5, Theorem 2] that if $V, W \in \mathbb{B}(\mathcal{H})$, for $z \in (0,1)$, then we get

$$ber^{\rho}(V+W) \leq \frac{1}{2}\|z(|V|^{\rho} + |W|^{\rho}) + (1-z)(|V^*|^{\rho} + |W^*|^{\rho})\|_{ber}, \quad (1.3)$$

for $\rho \geq 2$.

The article's originality or innovation stems from fresh estimation of the Berezin norm and Berezin radius of various types of some operators working on FHS. These estimates improve on the upper bounds of the Berezin numbers found in previous studies. In this section, we show useful Berezin radius inequalities for a FHS.

MAIN RESULTS

Let's prove the first theorem.

Theorem 1. If $V \in \mathbb{B}(\mathcal{H})$, $V \geq 0$, and f, g are nonnegative continouns functions on $[0,\infty)$ fulfilling $f(t)g(t) = t, (t \geq 0)$, then we get

$$ber^{\rho}(V) \leq \frac{1}{2}\|f^{2\rho}(|V|)g^{2\rho}(|V^*|)\|_{ber}^{1/2}, \quad (2.1)$$

for $\rho \geq 1$.

Proof. Let $\varsigma \in$ F be an arbitrary. Then, we have

$$\begin{aligned}
&\left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{2\rho} \\
&\leq \left\|f(|V|)\widehat{k_\varsigma}\right\|^{\rho}\left\|g(|V^*|)\widehat{k_\varsigma}\right\|^{\rho} \\
&= \langle f(|V|)\widehat{k_\varsigma}, f(|V|)\widehat{k_\varsigma}\rangle^{\rho} + \langle g(|V^*|)\widehat{k_\varsigma}, g(|V^*|)\widehat{k_\varsigma}\rangle^{\rho} \\
&\leq \langle f^2(|V|)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle^{\rho}\langle g^2(|V^*|)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle^{\rho} \\
&\leq \langle f^{2\rho}(|V|)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\langle g^{2\rho}(|V^*|)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle \\
&\leq \langle f^{2\rho}(|V|)g^{2\rho}(|V^*|)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle
\end{aligned}$$

where the fşrst inequality is obtained the in equality [7, Theorem 1] that; the fourth inequality is obtained the holder-Mccarthy inequality. Thus,

$$\left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{2\rho} \leq \langle f^{2\rho}(|V|)g^{2\rho}(|V^*|)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle.$$

Taking the supremum over $\varsigma \in$ F in the above inequality, i.e.,

$$sup_{\varsigma\in \mathrm{F}}\left(\left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{2\rho}\right) \leq sup_{\varsigma\in \mathrm{F}}\left(f^{2\rho}(|V|)g^{2\rho}(|V^*|)\widehat{k_\varsigma}, \widehat{k_\varsigma}\right),$$

we have

$$ber^{\rho}(V) \leq \frac{1}{2}\|f^{2\rho}(|V|)g^{2\rho}(|V^*|)\|_{ber}^{1/2}.$$

Then, it can be proved by using some arguments of the paper [4, Theorem 1] (which is omitted) that

$$\begin{aligned}
ber^{\rho}(V) &\leq \frac{1}{2}\|f^{2\rho}(|V|)g^{2\rho}(|V^*|)\|_{ber}^{\frac{1}{2}} \\
&\leq \frac{1}{2}\|f^{2\rho}(|V|) + g^{2\rho}(|V^*|)\|_{ber}. \quad (2.2)
\end{aligned}$$

Remark 1. *From the inequality (2.2) given above, the inequality (2.1) better than inequality (1.2).*

Our second conclusion extends the inequality (1.3) given in [5, Theorem 2], which offers a Berezin radius inequality.

Theorem 2. If $V, W \in \mathbb{B}(\mathcal{H})$, $V, W \geq 0$, and γ is any positive real number such that $0 < \gamma < 1$, then we have

$$\begin{aligned}
&ber^{\rho}(V+W) \\
&\leq 2^{\rho-1}\left\||V|^{\gamma\rho}|V^*|^{(1-\gamma)\rho} + |W|^{\gamma\rho}|W^*|^{(1-\gamma)\rho}\right\|_{ber},
\end{aligned} \quad (2.3)$$

for $\rho \geq 2$.

Proof. Let $\varsigma \in$ F be an arbitrary. Then, we have

$$\begin{aligned}
&\left|\langle (V+W)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{\rho} \\
&\leq \left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle + \langle W\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{\rho} \\
&= \left(\left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right| + \left|\langle W\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|\right)^{\rho} \\
&\leq 2^{\rho-1}\left(\left|\langle V\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{\rho} + \left|\langle W\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{\rho}\right) \\
&\leq 2^{\rho-1}\left(\langle |V|^{2\gamma}\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle^{\frac{\rho}{2}} + \langle |V^*|^{2(1-\gamma)}\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle^{\frac{\rho}{2}}\right) \\
&+ \left(\langle |W|^{2\gamma}\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle^{\frac{\rho}{2}} + \langle |W^*|^{2(1-\gamma)}\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle^{\frac{\rho}{2}}\right) \\
&\leq 2^{\rho-1}\langle\left(|V|^{\gamma\rho}|V^*|^{2(1-\gamma)} + |W|^{\gamma\rho}|W^*|^{2(1-\gamma)}\right)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle,
\end{aligned}$$

where the third inequality follows from the definition convex function , the fourth inequality from the inequality given [7, Corollary 7] and the fifth inequality follows from Hölder-McCaerthy inequality. Hence, taking the supremum over $\in$ F in the above inequality,

$$\begin{aligned}
&sup_{\varsigma\in \mathrm{F}}\left(\left|\langle (V+W)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right|^{\rho}\right) \\
&\leq sup_{\varsigma\in \mathrm{F}}\left(2^{\rho-1}\langle\left(|V|^{\gamma\rho}|V^*|^{2(1-\gamma)} + |W|^{\gamma\rho}|W^*|^{2(1-\gamma)}\right)\widehat{k_\varsigma}, \widehat{k_\varsigma}\rangle\right)
\end{aligned}$$

and

$ber^{\rho}(V+W)$
$\leq 2^{\rho-1}\left\|z\left(|V|^{\gamma\rho}|V^*|^{(1-\gamma)\rho}+|W|^{\gamma\rho}|W^*|^{(1-\gamma)\rho}\right)\right\|_{ber}.$

A basic and important inequality in functional analysis is the so-called Jensen type inequality, which assert that if $a, b \geq 0$ and $0 \leq \varsigma \leq 1$, then

$$a^{\varsigma}b^{1-\varsigma} \leq a\varsigma + (1-\varsigma)\mathrm{b} \leq (a^{\rho}\varsigma + (1-\varsigma)\mathrm{b}^{\rho})^{\frac{1}{\rho}} \qquad (2.4)$$

for $\rho \geq 1$.

Corollary 1. *(i) Using the inequality (2.4), we have*

$ber^{\rho}(V+W)$
$\leq 2^{\rho-1}\left\||V|^{\gamma\rho}|V^*|^{(1-\gamma)\rho}+|W|^{\gamma\rho}|W^*|^{(1-\gamma)\rho}\right\|_{ber}$
$\leq 2^{\rho-1}\|\gamma(|V|^{\rho}+|W|^{\rho})+(1-\gamma)(|V^*|^{\rho}+|W^*|^{\rho})\|_{ber}.$
(2.5)

(ii) If we take $\rho = 2$ and $\gamma = \frac{1}{2}$ in (2.3), then

$$ber^2(V+W) \leq 2^{\rho-1}\||V||V^*|+|W||W^*|\|_{ber} \qquad (2.6)$$

(iii) If we take $V = W$, then

$$ber(V) \leq \|V^2\|_{ber}^{\frac{1}{2}}.$$

(iv) If we take $W = 0$ and *and $\gamma = \frac{1}{2}$ in (2.5), then we have*

$$ber^{\rho}(V) \leq 2^{\rho-1}\left\||V|^{\rho/2}|V^*|^{\rho/2}\right\|_{ber} \leq 2^{\rho-1}\|V\|_{ber}^{\rho}$$

and

$$ber^{\rho}(V) \leq 2^{\rho-1}\left\||V|^{\frac{\rho}{2}}|V^*|^{\frac{\rho}{2}}\right\|_{ber}$$
$$\leq 2^{\rho-1}\left\||V|^{\rho/2}+|V^*|^{\rho/2}\right\|_{ber}.$$

Kaynakça

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