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***Research Article***

# Computational heuristics for Troesch's Problem in Plasma Physics

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***Abstract –*** Exponential Collocation Genetic Algorithm (ECGA) approach has been developed and analyzed for fast computation of hyperbolic sine nonlinear Troesch's problem which arises in the confinement of plasma. The governing equation is converted to an optimization problem by formulating the Fitness function in terms of an exponential basis. The problem is solved for three scenarios of Troesch's parameters of 0.5, 1, and 2 respectively. The stability of the solutions has been investigated for multiple independent runs. The results obtained in this work are in good agreement with the already published with enhanced stability and fast convergence. The developed technique is a simple and reliable method for the solution of hyperbolic sine nonlinear Troesch's problem.

*Keywords – Non-Linear Systems; Genetic Algorithm; Exponential Collocation; Singular Model; Troesch's Problem.*

## 1. Introduction

Troesch's Problem (TP) itself is a sensitive, unstable, hyperbolic nonlinear, and singular boundary value problem. Originally appear while examining the confinement of plasma by radiation pressure and in the theory of gas porous electrodes [ 1, 2]. The dimensionless form of TP is given as:

$$y''(x) - n\,Sinh(ny) = 0 \quad (1)$$

Constrained by, $$y(0) = 0 \qquad y(1) = 1 \quad (1a)$$

The hyperbolic sine nonlinearity and the presence of singularity make it highly difficult to solve analytically. The closed-form solution of Troesch's problem in terms of the Jacobian elliptic function was given by Roberts et al. 1976:

$$y(x) = \frac{2}{n}Sinh^{-1}\{\frac{y'(0)}{2}Sc(nx|1 - \frac{1}{4}(y'(0))^2)\} \quad (12)$$

Where $y'(0) = 2\sqrt{1-m}$, and constant m satisfies the solution of the transcendental equation

$$\frac{Sinh(\frac{n}{2})}{\sqrt{1-m}} = Sc(n|m) \quad (13)$$

Over there, the Jacobian elliptic function $sc(n|m)$is defined by $sc(n|m) = tan\phi$, where $\phi$,$n$, and $m$ are related by the integral

$$n = \int_0^{\phi} \frac{1}{\sqrt{1 - mSin^2\theta}}d\theta \quad (14)$$

Therefore, it's evident that there is a singularity located at a pole of $Sc(n|m)$ or nearly at [3]

$$x_s = \frac{1}{n}ln\,(\frac{8}{y'(0)} \quad (15)$$

This indicates that at $y'(0) > 8e^{-n}$, the singularity lies within the integration range. The presence of singularity in Troesch's equation makes the problem more challenging to be solved numerically. The problem with this equation is the convergence of the approximations on the boundaries as $n$

increases. The convergence problem to the exact solution at the upper boundary was only partially solved by using a combination of different numerical approaches. These methods are the sinc-collocation method [4], the modified homotopy perturbation method [ 5], a discontinuous Galerkin finite element method [6], smart nonstandard finite difference method [7], a hybrid asymptotic finite-element method [8], variational iteration method [9]. However, all the methods mentioned above are deterministic, serial in nature, and computationally costly.

The stochastic methods are relatively less exploited for the hyperbolic nonlinear systems having singularity. Artificial neural networks optimized with Genetic Algorithm (GA) have been tested for solving TP include [10-12]. However, due to the complexity of the neural network architecture and its weights, these stochastic solvers are hybrid with local search algorithms based on sequential quadratic programming for fast computation and to achieve quick convergence. A genetic Algorithm is a powerful stochastic search and optimization method based on the mechanics of natural selection. The idea of GA optimization is effectively applied to nonlinear differential equations for optimized solutions [13-16]. Through reviewing the literature, it can be seen that collocation method has not been applied to the Troesch's equation. Therefore, we are motivated to apply collocation method hybrid with the artificial intelligence techniques to determine the solution of this Troesch's boundary value problem. Furthermore, GA hybrid with exponential collocation can find an excellent solution for nonlinear differential equations with quick convergence [17] for complete input interval.

The present work aims at solving a nonlinear Troesch's problem by using an Exponential collocation Genetic Algorithm (ECGA). In this work, the nonlinear dynamics of Troesch's Problem are studied for the two cases of the step size h= of h=0.05 and 0.1 by taking the three scenarios of Troesch's parameter (n). The three scenarios have the value of n to be 0.5, 1, and 2 respectively. The prominent features of the proposed method are as follows:

- The exponential collocation is hybridized with GA to find the solution to Troesch's problem.
- Simulations have been conducted to find the best set of operators for the solution to Troesch's Problem.
- Exactness and stability analysis is conducted by using results of Mean Absolute Error (MAE), Root Mean Square Error (RMSE), and Standard Deviation (SD) respectively.
- Finally, compared the obtained solution to Troesch's Problem with the existing techniques.

The graphical abstract of the proposed method is shown in figure 1. The rest of the article is organized

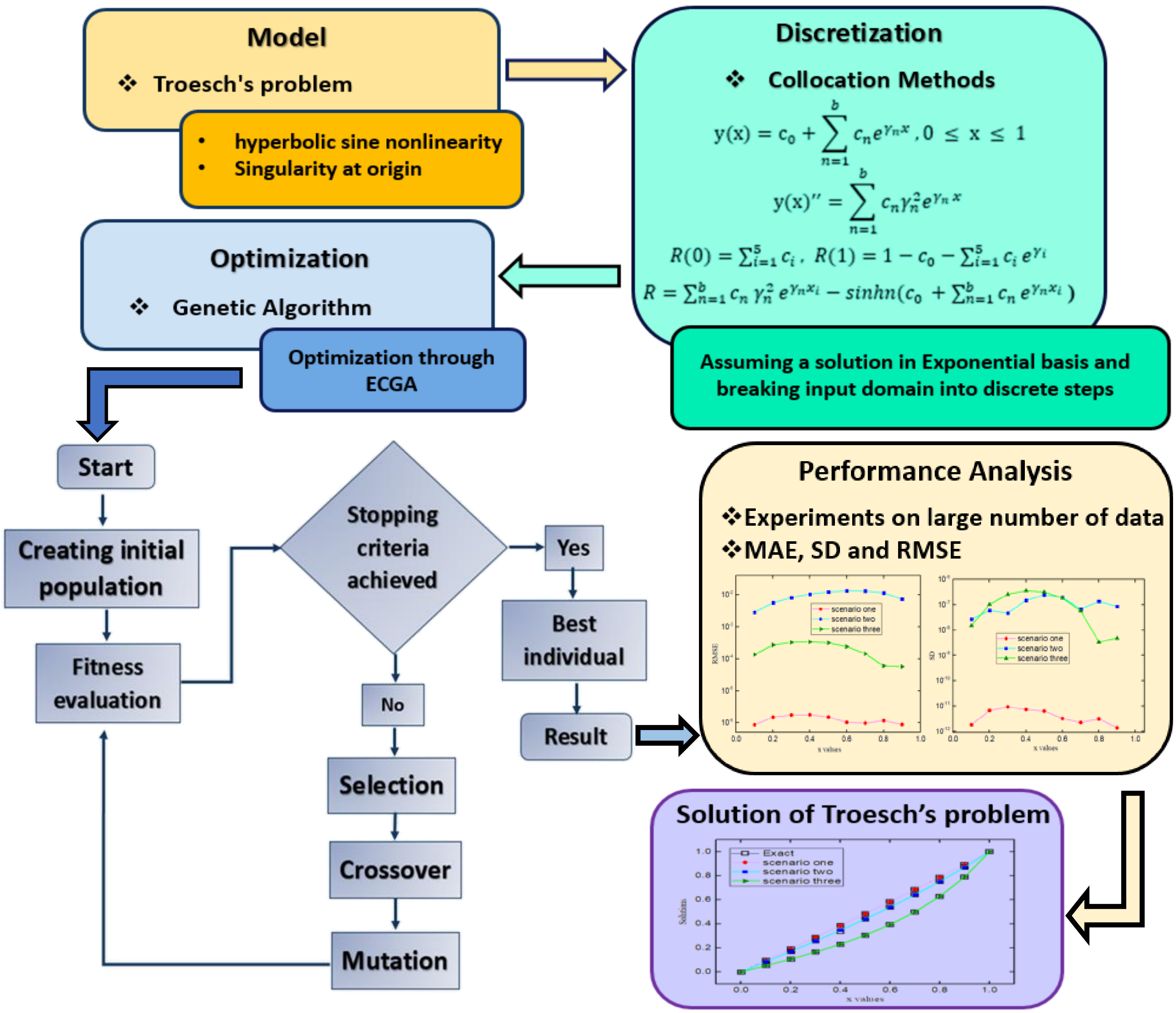



Fig. 1. The Proposed methodology for Troesch's Problem

as The proposed methodology is in section 2 and the results are discussed in section 3. Finally, the conclusion is given in section 4.

### 1.1.Troesch's Problem in Plasma Physics

Troesch's problem inherently is a sensitive, unstable, and highly non-linear two-point boundary value problem. The confinement of plasma by radiation pressure and in the theory of gas porous electrodes is given by Troesch's equation [Weibel, 1959].

$$\frac{1}{r}\frac{d}{dr}(r\frac{dE_0}{dr})+(w^2-\frac{e^2N}{M}-\frac{e^2n}{m})E_0 = 0 \quad (2)$$

$$\frac{1}{r}\frac{d}{dr}(rE_r)= e(N-n) \quad (3)$$

$$E_r = -\frac{du}{dr} \quad (4)$$

Where above equation represents the radial electrostatic field due to charge separation.

$$n(r) = n_0 \exp\,(\frac{eU}{KT} - \frac{e^2}{4mw^2kt} \quad (5)$$

$$N(r) = Nexp(-\frac{eU}{KT}-\frac{e^2E_0^2}{4Mw^2KT}) \quad (6)$$

Here, N and n are taken as ion and electron densities as a function of r. $E = E_r(r)$ is the electric field and $E_0(r)coswt)$ corresponds to the applied electric field plus the applied electric field due to the plasma current and T is the temperature; Furthermore, Ion and electron temperatures are assumed equal and constant. When $E_0$ is assumed to be negligibly small, then the system reduces to

$$\frac{dE}{dr} = N(r) - n(x) \quad (7)$$

$$E_r = -\frac{du}{dr} \quad (8)$$

$$n(x) = n_0 exp(\lambda U) \quad (9)$$

$$N(x) = N_0 exp(-\lambda U) \quad (10)$$

Substituting Equations (8) to (10) into Equation (7), a second-order nonlinear ordinary differential equation is obtained as:

$$\frac{d^2U}{dx^2}=N_0 exp((-\lambda U) n_0 exp(\lambda U) \quad (11)$$

Now, applying the simplifying assumption $N_0$ = $n_0$ $= N^*$and setting $U = -y$ Thus Equation (11) simplifies Troesch's problem. The equation in new variables can be written as Equation (1).

## 2. The proposed Methodology

Troesch's problem with the hyperbolic sine nonlinearity is converted to an optimization problem by proposing the solution in the form of exponential basis functions as [17]:

$$y(x) = c_0 + \sum_{n=1}^{b} c_n e^{\gamma_n x} , 0 \leq x \leq 1 \quad (16)$$

$$y(x)'' = \sum_{n=1}^{b} c_n \gamma_n^2 e^{\gamma_n x} \quad (17)$$

Where for $b = 5$ , the exponential basis set is given by $\{e^{\gamma_1 x}, e^{\gamma_2 x}, e^{\gamma_3 x}, e^{\gamma_4 x}, e^{\gamma_5 x}\}$ and $(c_0, c_1, c_2, c_3, c_4, c_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \text{ and } \gamma_5)$ are unknown coefficients respectively. We have discretized the input domains using the equally spaced collocation points.

### 2.1.Collocation Based Discretization

The discretization procedure involves the transformation of the proposed solution into an algebraic system of interrelated equations to formulate the fitness function. Equation (16) is converted into a set of algebraic equations in the form of exponential basis functions with constants $c_0, c_1, c_2, c_3, c_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4,$ and $\gamma_5$. The transformation is attained by putting the $y(x)$ *and* $y(x)''$in Eq. (16).

### 2.2. Fitness Function Formulation

The fitness function requirements are the residual equations that are made by the discretization procedure as discussed above together with the residual equations at the boundary points. The residual equation for discretized input domain is obtained from Equation (16), and it is given as

$$R = \sum_{n=1}^{b} c_n \gamma_n^2 e^{\gamma_n x_i} - sinhn(c_0 + \sum_{n=1}^{b} c_n e^{\gamma_n x_i} ) \quad (18)$$

Where $x_i = x_0 + h$ for different step sizes of 0.05, and 0.1, respectively. The residual equations at the boundary points are given as follows:

$$R(0) = \sum_{i=1}^{5} c_i \quad (19)$$

$$R(1) = 1 - c_0 - \sum_{i=1}^{5} c_i e^{\gamma_i} \quad (20)$$

The fitness function is formulated in terms of the overall residual error $O_R$. The norm of the whole input domain in terms of residues is used to define the overall residual function in the form of least square sense given as:

$$O_R = \sqrt{R(0)^2 + \sum_{i=h}^{i=1-h} \left(R(x_i)\right)^2 + R(1)^2} \quad (21)$$

In this way the original hyperbolic sine nonlinear problem is reduced to an equivalent optimization problem Now the task is the minimization of the fitness function $O_R$ so such that the errors of each equation decrease to achieve the zero value of the $O_R$. For that purpose, we will exploit the strength of the Genetic Algorithm as the global search optimization technique.

- **Case One:** In case one the input domain of Troesch's problem is discretized by taking 21 collocation points with the step size of h=0.05. Troesch's parameter (n) is taken to be n=0.5, 1, and 2.
- **Case Two:** In case two the input domain of Troesch's problem is discretized by taking 11 collocation points with the step size of h=0.1. Troesch's parameter (n) is taken to be n=0.5, 1, and 2.

### 2.4 Genetic Algorithm (GA)

Genetic Algorithm (GA) is a Global search algorithm centered on an evolutionary technique, where natural evolution and survival of the fittest are counterfeited to do a random search to get the best solution to a problem. Genetic Algorithm usually starts with an initial population of entities produced at random. [18] The basics of GA parameters include initial population size, crossover probability, mutation probability, and termination criteria. Simply, each candidate in the population represents a potential solution to the problem under consideration. Individuals evolve through successive iterations, called generations. During each generation, each candidate in the population is evaluated using some measure of fitness. Genotype is an individual's group of chromosomes, while phenotype is a set of values corresponding to a given genotype. The initial matrix $(p \times r)$ of $p$ number of chromosomes (solution set) of the GA is created by randomly bound numbers containing genes equal to several unknown variables $r$ in the operation of the residual function. Here each chromosome represents the discretization points of the collocation scheme. The basic operators of the genetic algorithm are election, crossover, and mutation.

- **Selection:** A process in which randomly selected individuals as parents based on their fitness evaluation. Selection is key to finding the best offspring from the initial population for crossover.
- **Crossover:** This function does the crossover between the two parent solutions according to the given fraction so that a new population can be formed.
- **Mutation:** This function induces diversity in the population by randomly generating new traits in the chromosomes.

## 3. Results and Discussion

In the current work, the Exponential Collocation Genetic Algorithm (ECGA) approach is applied to tackle the hyperbolic nonlinearity and singularity in Troesch's problem. The operators and the parameter settings for both cases are provided in Tables 1 and 2 respectively.

### 3.1. Simulations Results of Case One:

The approximate solutions obtained via ECGA for three scenarios together with the Mean Absolute Error (MAE) and the fitness value for 30 independent runs at the step size of h=0.05 are shown in Figure 2.

Table 1: Operators Used In ECGA For All Cases

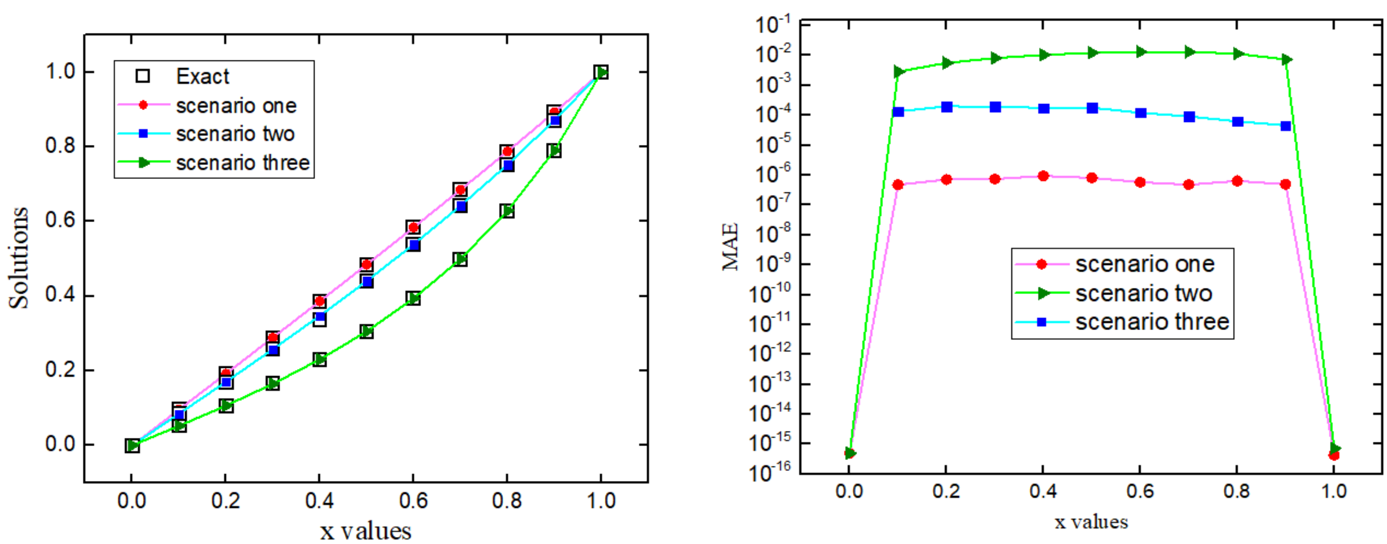
| Scheme | Operator | Setting |
|---|---|---|
| **ECGA** | Selection | Stochastic uniform |
| | Crossover | Heuristic |
| | Mutation | Adaptive feasible |

An essential characteristic of the optimization technique is that its result should maintain stability with an increasing number of runs. The ECGA executes a sequence of computations on the existing population at each iteration to create a new population. Each succeeding population is called a new generation. The fitness value of an individual is the value of the fitness function for that individual. The solution to Troesch's problem is showing moderate fitness as shown in Figure 2 (c). The fitness value is in the range of $10^{-3}$ to $10^{-1}$ and there is negligible fluctuation. The reproduction operators are wisely chosen to attain this fitness value. The reproduction operators are selected on a hit-and-trial basis, and as a final point, those are used for which the fitness value is minimum and fewer fluctuations occurred. The fitness value achieved at a population size of 200 for the three case scenarios varies from 0.0000199179 to 0.0000153761, 0.00409793 to 0.00307365, and 0.0369404 to 0.0569097 respectively. There is negligible fluctuation for the three scenarios that depict the stability of the ECGA for the hyperbolic nonlinear Troesch's problem.

Table 2: Parameters Setting for ECGA for all cases

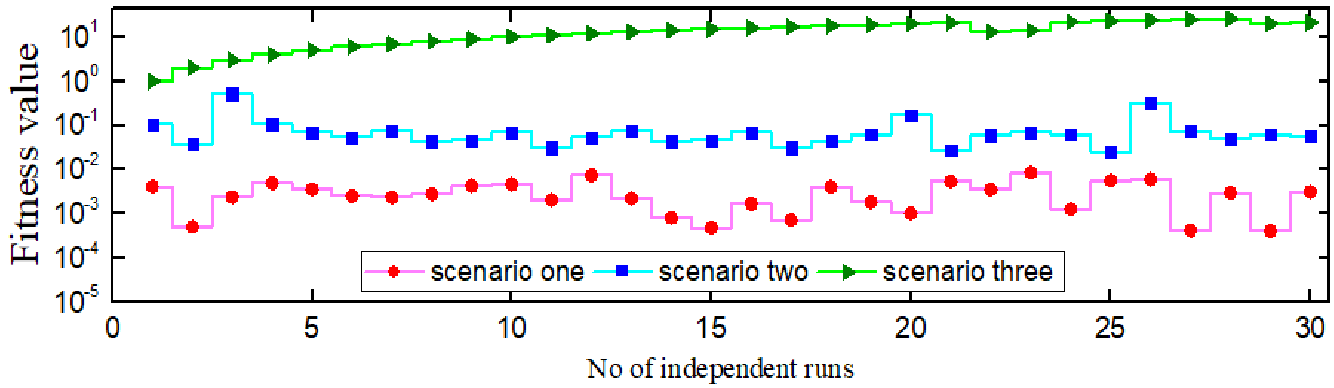
| **Parameter** | **Setting** |
|---|---|
| Population Size | 200 |
| Mutation fraction | 0.85 |
| Fitness limit | $10^{-15}$ |
| Elite count | 12 |
| No of Variables | 11 |
| Generations | 300 |
| Function Tolerance | $10^{-9}$ |
| Stall Generation limit | 100 |
| Nonlinear-constraint tolerance | $10^{-9}$ |
| Migration interval | 25 |
| Migration fraction | 0.2 |

The performance of the ECGA approach is precisely evaluated in terms of MAE. The small values of the MAE in the range of $10^{-6}$ to $10^{-2}$ as shown in Figure 2(b) precisely depicts the performance of the proposed ECGA for the hyperbolic nonlinearity. In Table 3 the performance of the ECGA solution is assessed by comparing the results with the mean solution of 30 independent runs and the exact solution values. The convergence is achieved statistically based on the number of runs scored from data of 30 successful runs.




(a)

(b)




(c)

Fig. 2. (a) Solution of Troesch's problem case one, (b) Mean Absolute Error (MAE), (c) Fitness value for 30 independent runs

Table 3: Result of the exact solution of case one, GA solution, and GA mean of 30 independent runs for all case scenarios for h=0.05

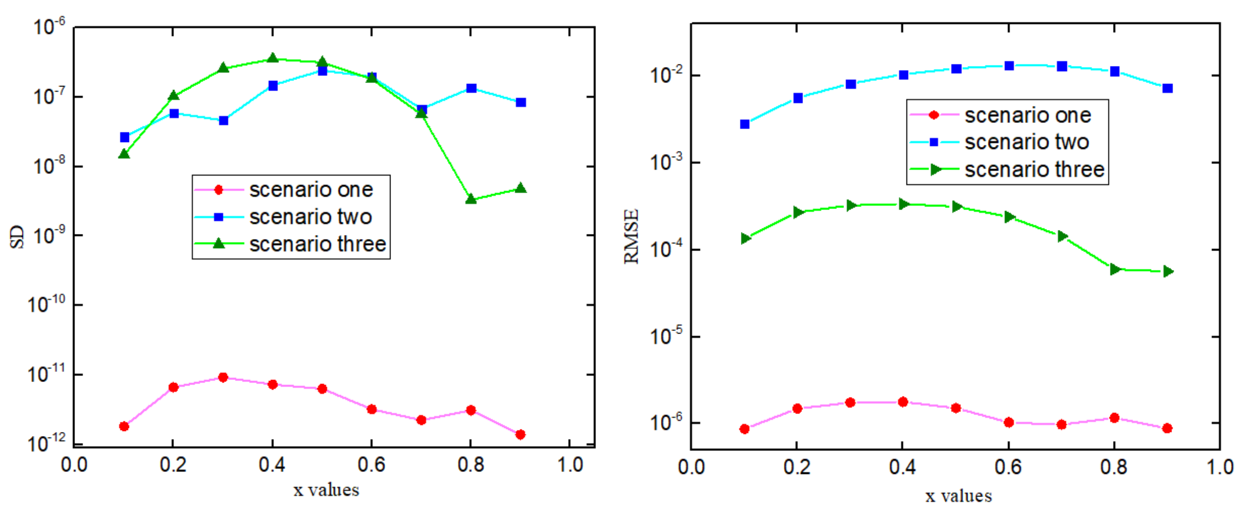
| x values | n=0.5 | | | n=1 | | | n=2 | | |
|---|---|---|---|---|---|---|---|---|---|
| | Exact Solution | ECGA Solution | ECGA Mean Solution | Exact Solution | ECGA Solution | ECGA Mean Solution | Exact Solution | ECGA Solution | ECGA Mean Solution |
| **0.0** | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 1E-15 | 1.7E-16 | 0.00000 | 0.00000 | 0.00000 |
| **0.1** | 0.09594 | 0.09594 | 0.09594 | 0.08466 | 0.08465 | 0.08465 | 0.05220 | 0.05213 | 0.05218 |
| **0.2** | 0.19212 | 0.19212 | 0.19212 | 0.17017 | 0.17016 | 0.17016 | 0.10651 | 0.10628 | 0.10653 |
| **0.3** | 0.28879 | 0.28879 | 0.28879 | 0.25739 | 0.25738 | 0.25739 | 0.16514 | 0.16481 | 0.16515 |
| **0.4** | 0.38618 | 0.38618 | 0.38618 | 0.33673 | 0.34722 | 0.34722 | 0.23052 | 0.23025 | 0.23051 |
| **0.5** | 0.48454 | 0.48454 | 0.48454 | 0.44059 | 0.44061 | 0.44061 | 0.30550 | 0.30544 | 0.30549 |
| **0.6** | 0.58413 | 0.58413 | 0.58413 | 0.53853 | 0.53854 | 0.53854 | 0.39356 | 0.39374 | 0.39358 |
| **0.7** | 0.68520 | 0.68520 | 0.68520 | 0.64212 | 0.64213 | 0.64213 | 0.49917 | 0.49943 | 0.49923 |
| **0.8** | 0.78801 | 0.78801 | 0.78801 | 0.75260 | 0.75260 | 0.75260 | 0.62846 | 0.62855 | 0.62850 |
| **0.9** | 0.89285 | 0.89285 | 0.89285 | 0.87136 | 0.87135 | 0.87135 | 0.79049 | 0.79036 | 0.79045 |
| **1.0** | 1.00000 | 1.00000 | 1.00000 | 1.0000 | 1.00000 | 1.00000 | 1.0000 | 1.00000 | 1.00000 |

The exactitude and stability of the designed ECGA are further validated in terms of the Root Mean Square Error (RMSE) and Standard Deviation (SD), as given in Table 4. The small values of these statistical performance indicators show the validity of the ECGA method.

Table 4: Results of statistical operators based on 30 independent runs of ECGA case one for three scenarios of n=0.5, 1 and 2

| x values | n=0.5 | | | n=1 | | | n=2 | | |
|---|---|---|---|---|---|---|---|---|---|
| | **MAE** | **RMSE** | **SD** | **MAE** | **RMSE** | **SD** | **MAE** | **RMSE** | **SD** |
| **0.0** | 5E-16 | 1.2E-15 | 5.21E-30 | 5E-16 | 9.4E-16 | 1.92E-30 | 0 | 1.25E-15 | 5.21E-30 |
| **0.1** | 4.74E-07 | 8.69E-07 | 1.82E-12 | 0.00286 | 0.00285 | 2.15E-08 | 0.0001 | 8.69E-07 | 1.82E-12 |
| **0.2** | 7.07E-07 | 1.49E-06 | 6.67E-12 | 0.005636 | 0.00563 | 5.34E-08 | 0.0002 | 1.49E-06 | 6.67E-12 |
| **0.3** | 7.38E-07 | 1.75E-06 | 9.27E-12 | 0.008226 | 0.00822 | 5.83E-08 | 0.0001 | 1.75E-06 | 9.27E-12 |
| **0.4** | 9.34E-07 | 1.78E-06 | 7.32E-12 | 0.010497 | 0.01049 | 1.37E-07 | 0.0001 | 1.78E-06 | 7.32E-12 |
| **0.5** | 8.07E-07 | 1.51E-06 | 6.31E-12 | 0.012264 | 0.01226 | 2.04E-07 | 0.0001 | 1.51E-06 | 6.31E-12 |
| **0.6** | 5.79E-07 | 1.03E-06 | 3.21E-12 | 0.01327 | 0.01326 | 1.57E-07 | 0.0001 | 1.03E-06 | 3.21E-12 |
| **0.7** | 4.77E-07 | 9.75E-07 | 2.24E-12 | 0.01316 | 0.01316 | 8.96E-08 | 9.1E-05 | 9.75E-07 | 2.24E-12 |
| **0.8** | 6.39E-07 | 1.17E-06 | 3.1E-12 | 0.011435 | 0.01143 | 1.44E-07 | 6.2E-05 | 1.17E-06 | 3.1E-12 |
| **0.9** | 4.94E-07 | 7.9E-07 | 1.24E-12 | 0.007386 | 0.00738 | 7.47E-08 | 4.5E-05 | 7.99E-07 | 1.24E-12 |
| **1.0** | 4.33E-16 | 9.82E-16 | 2.97E-30 | 7.33E-16 | 2.5E-15 | 3.02E-29 | 0 | 9.82E-16 | 2.97E-30 |

Furthermore, there are minute fluctuations in both the RMSE and SD values as shown in Figure 3. The RMSE values vary from a minimum $10^{-6}$ to a maximum $10^{-2}$ indicating the stability of the designed scheme for various independent runs.




(a) (b)

Fig. 3. (a) Standard Deviation (SD), (b) Root Mean Square Error (RMSE) for three scenarios for case one

### 3.2. Simulations Results of Case Two:

The approximate solutions obtained via ECGA for three scenarios together with the Mean Absolute Error (MAE) and the fitness value for 30 independent runs at the step size of h=0.05 are shown in Figure 4. The fitness values as shown in Figure 4(c) vary in the range of 0.0000667153 to 0.000050541, 0.000561028 to 0.00524705, and 0.0177713 to 0.0500459 for the three scenarios respectively. The fitness values are smaller for case two than for the previous case due to the lesser number of collocation points. Furthermore, the fitness values show negligible variation for all three scenarios of Troesch's parameter. The MAE values are found to be overlapping in the order of $10^{-4}$ for the three scenarios as shown in Figure 4 (b).

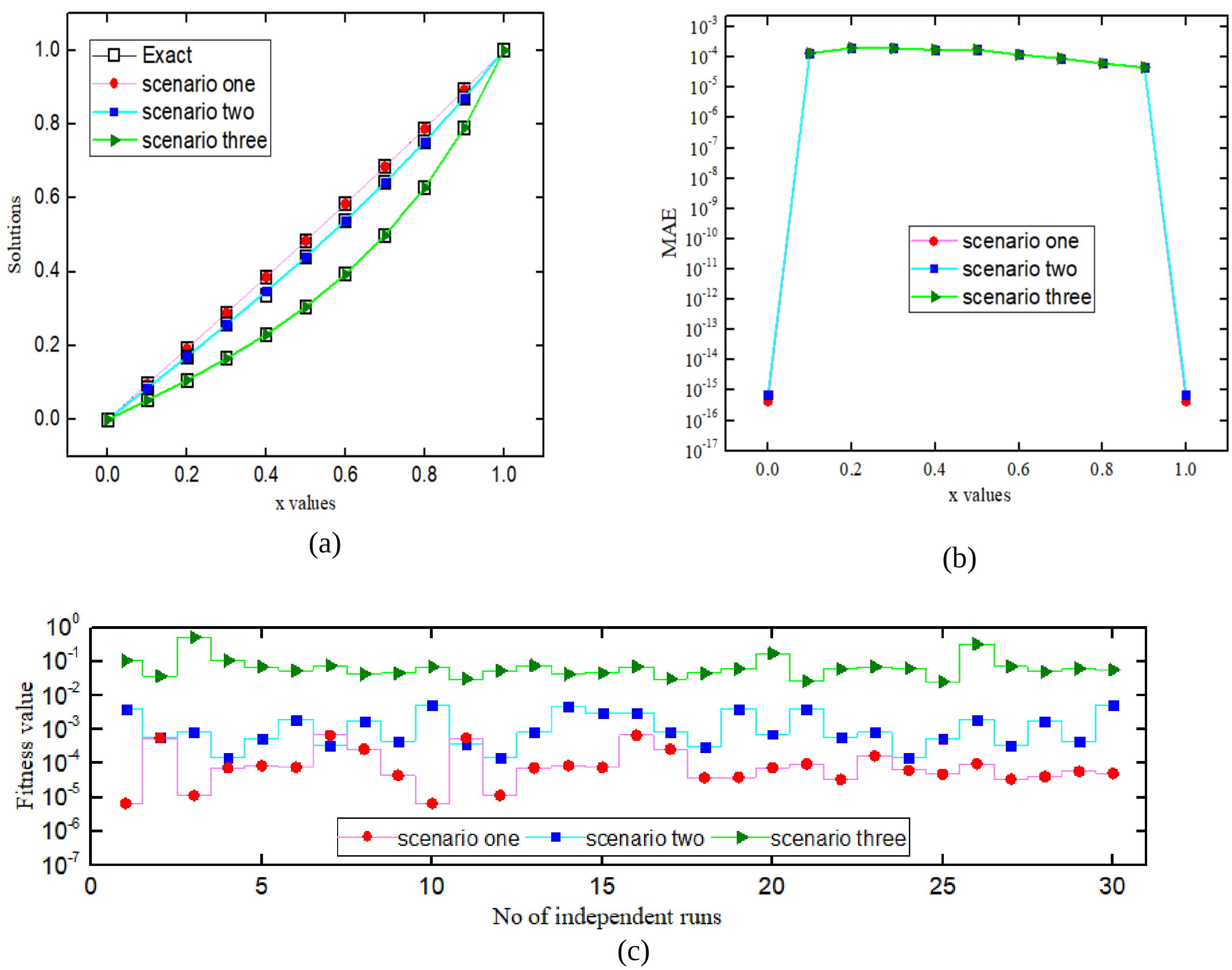



Fig.4. (a) Solution of Troesch's problem case 2 (h=0.1), (b) Mean Absolute Error (MAE), (c) Fitness value for 30 independent runs for case two

The results ECGA mean solution for the 30 runs for case two are given in Table 6. It is observed that the values of ECGA and ECGA mean are quite close to the exact values indicating the higher stability of the proposed approach with an accuracy of up to five decimal places. The results and statistical analysis for the 30 runs of ECGA are given in Table 7. For the values of x=0 and x=1, the values of MAE, RMSE, and SD are very small due to the exact value of the solution being known at boundary points. MAE values are obtained to be 4.79E-7 to 4.99E-7, 0.0028 to 0.0073, 0.0001 to 4.59E-5 for n=0.5, 1, and 2 respectively. Similarly, RMSE and SD values are obtained in the range of 8.69E-7 to 8.82E-7, 0.0028 to 0.0073, 0.0001 to 5.67E-5and 1.8E-12 to 1.24E-12, 2.69E-8 to 8.51E-08, and 1.48E-08 to 4.8E-09 as shown in Figure 5.

Table 6. Result of the exact solution, GA solution, and GA mean solution of 30 independent runs for case two

| x values | n=0.5 | | | n=1 | | | n=2 | | |
|---|---|---|---|---|---|---|---|---|---|
| | **Exact solution** | **ECGA solution** | **ECGA Mean solution** | **Exact solution** | **ECGA solution** | **ECGA Mean solution** | **Exact solution** | **ECGA solution** | **ECGA Mean solution** |
| **0.0** | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 1E-15 | 1.66E-16 | 0.00000 | 0.00000 | 0.00000 |
| **0.1** | 0.09594 | 0.09594 | 0.09594 | 0.08466 | 0.08464 | 0.08465 | 0.05220 | 0.05226 | 0.05218 |
| **0.2** | 0.19212 | 0.19212 | 0.19212 | 0.17017 | 0.17015 | 0.170166 | 0.10651 | 0.10658 | 0.10653 |
| **0.3** | 0.28879 | 0.28879 | 0.28879 | 0.25739 | 0.25738 | 0.257393 | 0.16514 | 0.16516 | 0.16515 |
| **0.4** | 0.38618 | 0.38618 | 0.38618 | 0.33673 | 0.34723 | 0.347229 | 0.23052 | 0.23048 | 0.23051 |
| **0.5** | 0.48454 | 0.48454 | 0.4845 | 0.44059 | 0.44061 | 0.440611 | 0.30550 | 0.30543 | 0.30549 |
| **0.6** | 0.58413 | 0.58413 | 0.58413 | 0.53853 | 0.53855 | 0.53854 | 0.39356 | 0.39351 | 0.39358 |
| **0.7** | 0.68520 | 0.68520 | 0.68520 | 0.64212 | 0.64213 | 0.64213 | 0.49917 | 0.49917 | 0.49923 |
| **0.8** | 0.78801 | 0.78801 | 0.78801 | 0.75260 | 0.75260 | 0.75260 | 0.62846 | 0.62848 | 0.62850 |
| **0.9** | 0.89285 | 0.89285 | 0.89285 | 0.87136 | 0.87135 | 0.87135 | 0.79049 | 0.79047 | 0.79045 |
| **1.0** | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |

Table 7. Results of statistical operators based on 30 independent runs of ECGA for case two

| x values | n=0.5 | | | n=1 | | | n=2 | | |
|---|---|---|---|---|---|---|---|---|---|
| | **MAE** | **RMSE** | **SD** | **MAE** | **RMSE** | **SD** | **MAE** | **RMSE** | **SD** |
| **0.0** | 5E-16 | 1.2E-15 | 5.2E-30 | 5E-16 | 1.2E-15 | 5.2E-30 | 0 | 0 | 0 |
| **0.1** | 4.79E-7 | 8.69E-7 | 1.8E-12 | 0.0028 | 0.0028 | 2.69E-8 | 0.0001 | 0.0001 | 1.48E-8 |
| **0.2** | 7.09E-07 | 1.49E-06 | 6.67E-12 | 0.0056 | 0.0056 | 5.91E-08 | 0.0002 | 0.0002 | 1.03E-07 |
| **0.3** | 7.39E-07 | 1.75E-06 | 9.27E-12 | 0.0082 | 0.0082 | 4.65E-08 | 0.0001 | 0.0003 | 2.57E-07 |
| **0.4** | 9.33E-07 | 1.78E-06 | 7.32E-12 | 0.0104 | 0.0104 | 1.49E-07 | 0.0001 | 0.0003 | 3.55E-07 |
| **0.5** | 8.06E-07 | 1.51E-06 | 6.31E-12 | 0.0122 | 0.0122 | 2.43E-07 | 0.0001 | 0.0003 | 3.17E-07 |
| **0.6** | 5.71E-07 | 1.03E-06 | 3.21E-12 | 0.0132 | 0.0132 | 1.96E-07 | 0.0001 | 0.0002 | 1.83E-07 |
| **0.7** | 4.78E-07 | 9.75E-07 | 2.24E-12 | 0.0131 | 0.0131 | 6.79E-08 | 9.18E-05 | 0.0001 | 5.65E-08 |
| **0.8** | 6.31E-07 | 1.17E-06 | 3.1E-12 | 0.0114 | 0.0114 | 1.35E-07 | 6.23E-05 | 6E-05 | 3.33E-09 |
| **0.9** | 4.99E-07 | 8.82E-07 | 1.24E-12 | 0.0073 | 0.0073 | 8.51E-08 | 4.59E-05 | 5.67E-05 | 4.8E-09 |
| **1.0** | 4.33E-16 | 9.82E-16 | 2.9E-30 | 7.3E-16 | 0 | 4E-30 | 0 | 0 | 0 |

The optimized value of the constants ($c_0, c_1, c_2, c_3, c_4, c_5, \gamma_1, \gamma_2, \gamma_3\ \gamma_4\ \gamma_5$) achieved at the best fitness value for both cases are shown in figure 6. It can be seen that the values of the constants are smaller for case two with a lesser number of collocation points. The small values of the constants indicate the small solution space for the less number of collocation points is small as compared to the larger collocation points. Therefore, the ECGA computation is fast and converges more quickly to the solution for case two with the smaller fitness value.

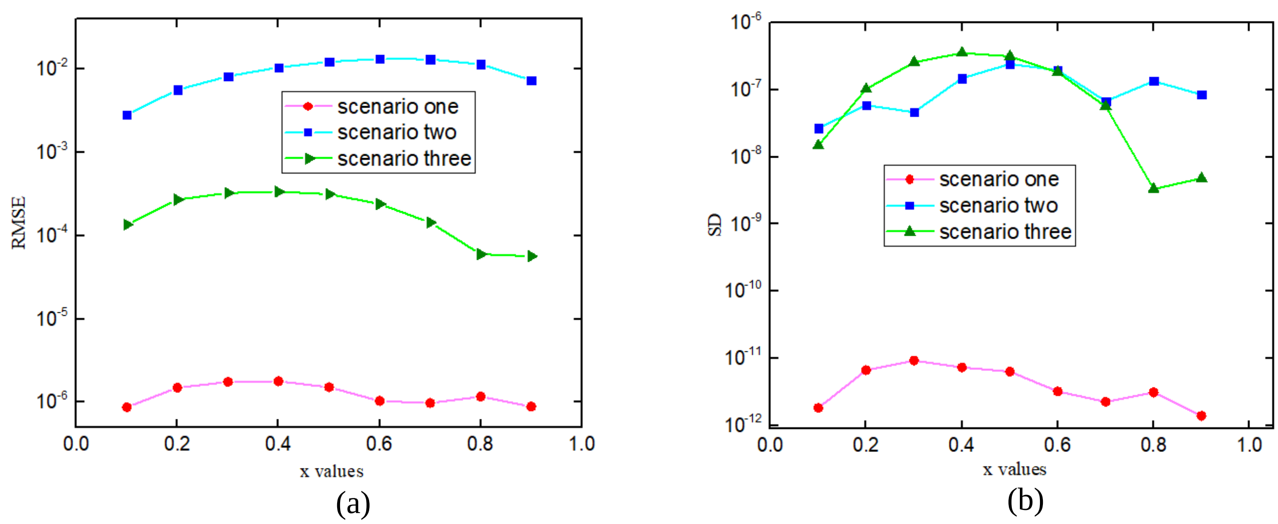
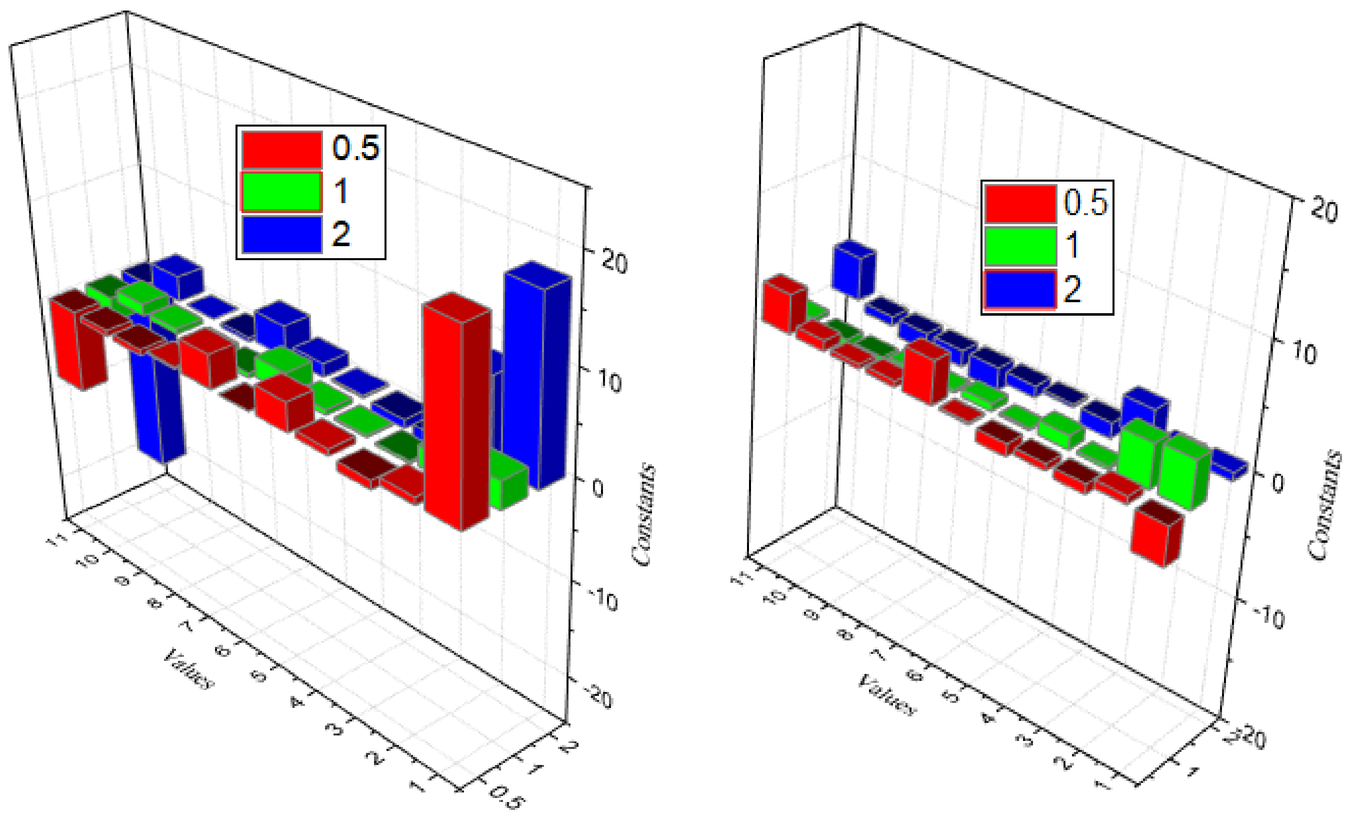



Fig. 5. (a) Standard Deviation (SD), (b) Root Mean Square Error (RMSE) for three scenarios of case 2 (h=0.1)




(a)

(b)

Fig. 6. Constant values achieved for best fitness of ECGA for (a) case one (b) case two for all the scenarios of Troesch's parameter

The performance of the collocation genetic algorithm approach is precisely evaluated. The comparison with the other method is shown in Table 5. Here we compare our results with the deterministic and stochastic methods, including the Decomposition Method (DM) [19], Modified Homotopy perturbation method (MHPM) [5], and Unsupervised Neural Network (UNN) hybrid with GA [11] at two different step sizes. The deterministic methods do not yield a good approximation because the nonlinear term will not be analytic for Troesch's parameter greater than n=1. Furthermore, these methods require a large number of series terms for an accurate solution at the expense of computational cost. Whereas the stochastic previously applied to solve Troesch's problem is hybridized with the interior point method (IPM), and pattern search (PS) to achieve quick convergence to an accurate solution. Whereas, the ECGA approach is simple and precise in the entire input domain without the aid of local search algorithms with fast computation and quick convergence to the optimal solution.

Table 5. Comparison of the results from different methods for Troesch's problem n=0.5

| x values | Exact Solution | DM [19] Solution | MHPM [5] Solution | UNN [11] Solution | ECGA Solution |
|---|---|---|---|---|---|
| **0.1** | 0.0959443493 | 0.0959383534 | 0.09593956 | 0.095944354 | 0.095944349 |
| **0.2** | 0.1921287477 | 0.1921180592 | 0.19211932 | 0.192128746 | 0.19212874 |
| **0.3** | 0.2887944009 | 0.2887803297 | 0.28878069 | 0.288794397 | 0.28879440 |
| **0.4** | 0.3861848464 | 0.3861687095 | 0.38616754 | 0.386184855 | 0.386184846 |
| **0.5** | 0.4845471647 | 0.4845302901 | 0.4845274183 | 0.4845471933 | 0.484547165 |
| **0.6** | 0.5841332484 | 0.5841169798 | 0.5841127822 | 0.584133285 | 0.584133248 |
| **0.7** | 0.6852011483 | 0.6851868451 | 0.6851822495 | 0.6852011710 | 0.685201148 |
| **0.8** | 0.7880165227 | 0.7880055691 | 0.7880018367 | 0.7880165165 | 0.788016523 |
| **0.9** | 0.8928542161 | 0.8928480234 | 0.8928462193 | 0.8928541967 | 0.892854216 |
| **1.0** | 1.0000000000 | 0.9999999988 | 1.0000000000 | 1.0000000000 | 1.000000000 |

## 4. Conclusions

We have developed an Exponential collocation Genetic Algorithm (ECGA) technique to solve nonlinear Troesch's problem. We have compared obtained results with the exact and numerical solution already given. The following conclusions can be drawn from the current study.

- Fitness values achieved are in the range of $10^{-3}$ to $10^{-6}$ for the two case scenarios.
- A substantially large number of runs to verify that the solution does not deviate or behave unusually. Our results have shown that the ECGA solutions are very stable and even after multiple runs, generate good stability.
- The statistical error indices are of the order of $10^{-6}$ and have slight variations.
- The research work shows the potent ability of the ECGA approach to search for the optimal solution from amongst a very large set of candidate solutions. Therefore, Genetic Algorithms are a good choice for solutions to highly nonlinear problems.

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