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Research Article

The Eigenvalues of Three-Interval Sturm-Liouville Problems with Additional Impulsive Conditions

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Abstract – The main goal of this work is to study some properties of eigenvalues and corresponding eigenfunctions of a new type boundary value problems consisting of three-interval Sturm-Liouville equation

$$Lu := -u''(x) + q(x)u = \lambda u, \ \ x \in [a,\xi_1) \cup (\xi_1,\xi_2) \cup (\xi_2,b]$$

subject to parameter-dependent boundary conditions (which we call parameter-dependent periodic boundary conditions), given by

$$u(a) = \alpha u(b), \ \ u'(a) = \beta u'(b)$$

and additional impulsive conditions at the common endpoints $\xi_1, \ \xi_2$, given by

$$u(\xi_i - 0) = \alpha u(\xi_i + 0), \ u'(\xi_i + 0) = \beta u'(\xi_i + 0)\ , \ \text{i=1,2}$$

where $q(x)$ is real-valued functions which continuous on each of the intervals $[a,\xi_1)$, (ξ_1,ξ_2) and $(\xi_2,b]$ and has a finite one-hand limits $q(\xi_i \pm 0) = \lim_{x\to\xi_i^\pm} q(x)$, α≠0 and β≠0 is a real parameter, λ is a complex eigenvalue parameter.

In the special case when α= β=1, $q(\xi_i + 0) = q(\xi_i - 0)(i = 1,2)$ the problem under consideration is reduced to classical periodic Sturm-Liouville problems, so the results obtained in this paper extend and generalize the corresponding classicals results. Note that the problem under consideration is not selfadjoint in the classical Hilbert space of square integrable functions. By using a new approaches we obtained some important properties of eigenvalues and eigenfunctions.

Keywords – Sturm-Liouville Problems, Impulsive Conditions, Eigenvalue, Eigenfunction.

I. INTRODUCTION

Many important problems appearing in physics and other branches of natural science are described by second order ordinary lineer differential equations, which can be presented in a self-adjoint form known as Sturm-Liouville equation (SLE). SLEs fisrt appeared in the context of the separation of variables method for partial differential equations of various types. This and related methods continue to generate Sturm-Liouville problems that model phenomena such as seismic behaviour, sonar propogation in water stratified by different densities, heat and mass transfer, large-scale waves in the atmosphere, etc. (see, for example, [4], [7], [12]).

The history of periodic spectral theory begins with studies of Sturm and Liouville on the spectral analysis odf second order differential equations with some boundary conditions, now called Sturm-Liouville problems (SLPs).

The spectral analysis of SLPs required the study of various spectral properties, such as discretenes of the spectrum, asymptotic behaviour of eigenvalues and corresponding eigenfunctions, completeness of eigenfunctions, location of zeros of the eigenfunctions, oscillation of the solutions so on. There is now an extensive literature on Sturm-Liouville theory and its applications (see, for example, [1], [8]-[10], [12]). In recent years, SLEs with periodic boundary conditions have become an important area of applied and theoritical mathematics, because the needs of modern physics and technology (see, for example, [2], [3], [5], [6]) and refences, cited therein.

In this work we will consider Sturm-Liouville equation defined on three non-intersecting intervals together with paramater-dependent periodic boundary conditions and additional transmission conditions specified at the internal points of interaction. This type of problem will be called three-interval Sturm-Liouville boundary-transmission problem (3-SLBTP, for short). Note that boundary value problems with additional impulsive conditions often arise in various fields of physics and tecnology. For example, in electrostatics and magnetostatics, in free oscillations of Earth, in heat transfer through an infinitely conducting layer, in hydraulic fracturing etc. (see, for example, [4], [7], [13]).

II. MAIN RESULTS

Consider a Sturm-Liouville equation

$$Lu := -u''(x) + q(x)u(x) = \lambda u(x), \qquad (1)$$

$x \in [a,\xi_1) \cup (\xi_1,\xi_2) \cup (\xi_2,b]$, defined on three non-intersecting intervals $[a,\xi_1), (\xi_1,\xi_2)$ and $(\xi_2,b]$ subject to parameter-dependent boundary conditions given by

$$\mathrm{u}(a) = \alpha \mathrm{u}(\mathrm{b}), \quad \mathrm{u}'(a) = \beta \mathrm{u}'(\mathrm{b}) \qquad (2)$$

(so called parameter-dependent periodic boundary conditions), and additional impulsive conditions at the points of interaction $x = \xi_1$ and $x = \xi_2$ given by

$$\mathrm{u}(\xi_1 - 0) = \alpha \mathrm{u}(\xi_1 - 0) \qquad (3)$$

$$\mathrm{u}'(\xi_1 + 0) = \beta \mathrm{u}'(\xi_1 + 0) \qquad (4)$$

$$\mathrm{u}(\xi_2 - 0) = \alpha \mathrm{u}(\xi_2 - 0) \qquad (5)$$

$$\mathrm{u}'(\xi_2 + 0) = \beta \mathrm{u}'(\xi_2 + 0) \qquad (6)$$

where $q(x)$ is real-valued function which continuous on each of the intervals $\Omega_1 = [a,\xi_1)$, $\Omega_2 = (\xi_1,\xi_2)$ and $\Omega_3 = (\xi_2,b]$ and have a finite limits $q(\xi_i \pm 0) = \lim_{x\to\xi_i\pm} q(x)$, α≠0 and β≠0 are real parameters, λ is a complex spectral parameter.

Let us denote by $\oplus_{i=1}^{3} C^k(\Omega_i)$ the function space consisting of all functions $f{:}\cup_{i=1}^{3} \Omega_i \to R$, that are continuously differetiable up to K-th order the each intervas Ω_i, and by $\cup_{i=1}^{3} C^k\left(\overline{\Omega_i}\right)$ the function space consisting of all functions $f{:}\cup_{i=1}^{3} \Omega_i \to R$, that are continuously differetiable up to K-th order an each intervas Ω_i and have finite limit values $f^{(s)}(\xi_i \pm 0)$ for $s = 1,2,\dots,K$.

Lemma 1. If

$$u, v \in \left(\oplus_{i=1}^{3} C^2(\Omega_i)\right) \cap \left(\oplus_{i=1}^{3} C^1\left(\overline{\Omega_i}\right)\right)$$

then:

$$\int_{a}^{\xi_1-0} (\upsilon L(u) - uL(\upsilon))dx + \int_{\xi_1+0}^{\xi_2-0} (\upsilon L(u) - uL(\upsilon))dx + \int_{\xi_2+0}^{b} (\upsilon L(u) - uL(\upsilon))\, dx$$

$$= W(u,\upsilon;x)|_a^b - JW(u,v)|_{\xi_1} - JW(u,v)|_{\xi_2} \quad (7)$$

where $Jf|_{x_0} = f(x_0+0) - f(x_0-0)$ is the jump function.

Proof. Integrating by parts over the intervals Ω_i (i=1,2) we have

$$\sum_{i=1}^{3}\int_{\Omega_i} \upsilon(Lu)dx = \sum_{i=1}^{3}\int_{\Omega_i}(\upsilon' u' + quv)dx - u'v|_a^b - J(u'v)|_{\xi_1} - J(u'v)|_{\xi_2} \quad (8)$$

Now, by symmetri we see that

$$\sum_{i=1}^{3}\int_{\Omega_i} u(L\upsilon)dx = \sum_{i=1}^{3}\int_{\Omega_i}(u'\upsilon' + quv)dx - \upsilon' u|_a^b - J(uv')|_{\xi_1} - J(uv')|_{\xi_2} \quad (9)$$

Subtracting (9) from (8) we obtain (7).

Definiton 2. A number $\lambda \in C$ is said to be an eigenvalue for the three-interval Sturm-Liouville boundary-transmission problem (1)-(6) if there exists a function

$$u \in \left(\oplus_{i=1}^{3} C^2(\Omega_i)\right) \cap \left(\oplus_{i=1}^{3} C^1(\overline{\Omega_i})\right)$$

such that u is not identically zero and satisfies the three-interval Sturm-Liouville equation (1) and boundary-transmission conditions (2)-(6). The solution $u(x)$ is called an eigenfunction corresponding to the eigenvalue and the pair (λ, u) is called an eigenpair.

Theorem 3. Let $(\lambda, u(x))$ be an eigenpair of the 3-SLBTP (1)-(6). If αβ=1, then the eigenvalue λ is real.

Proof. Since α, β are real numbers and $q(x)$ is real-valued function we get the following equalities upon taking complex conjugate of Sturm-Liouville equation (1) and boundary-transmission conditions (2)-(6)

$$L\bar{u} = \bar{\lambda}\bar{u}, \quad (10)$$

$$\bar{u}(a) = \alpha\bar{u}(b), \quad \bar{u}'(a) = \beta\bar{u}'(b) \quad (11)$$

$$\bar{u}(\xi_1 - 0) = \alpha\bar{u}(\xi_1 - 0) \quad (12)$$

$$\bar{u}'(\xi_1 + 0) = \beta\bar{u}'(\xi_1 + 0) \quad (13)$$

$$\bar{u}(\xi_2 - 0) = \alpha\bar{u}(\xi_2 - 0) \quad (14)$$

$$\bar{u}'(\xi_2 + 0) = \beta\bar{u}'(\xi_2 + 0) \quad (15)$$

The equalities (10)-(15) means that $(\bar{\lambda}, \bar{u}(x))$ is also an eigenpair for the 3-SLBTP (1)-(6).
From (1) it follows that

$$\sum_{i=1}^{3}\int_{\Omega_i} \bar{u}\, Lu dx = \lambda \sum_{i=1}^{3}\int_{\Omega_i} |u(x)|^2 dx \quad (16)$$

Similarly from (10) we have

$$\sum_{i=1}^{3}\int_{\Omega_i} uL\bar{u}\, dx = \bar{\lambda}\sum_{i=1}^{3}\int_{\Omega_i} |u(x)|^2 dx \quad (17)$$

Subtracting (16) from (17) yields

$$\sum_{i=1}^{3}\int_{\Omega_i} \left(uL(\bar{u}) - \bar{u}L(u)\right)dx = \left(\lambda - \bar{\lambda}\right) \oplus_{i=1}^{3} \int_{\Omega_i} |u(x)|^2 dx \quad (18)$$

Applying Lemma 1 we get

$$\left(\lambda - \bar{\lambda}\right)\sum_{i=1}^{3}\int_{\Omega_i}|u(x)|^2 dx = W(u,\bar{u};x)|_a^b - JW(u,\bar{u};x)|_{\xi_1} - JW(u,\bar{u};x)|_{\xi_2} \quad (19)$$

From (2) and (11)

$$W(u,\bar{u};a) = u(a)\overline{u'}(a) - u'(a)\bar{u}(a)$$

$$= \alpha\beta\left(u(b)\overline{u'}(b) - u'(b)\bar{u}(b)\right)$$

$$=W(u,\bar{u};b) \qquad (20)$$

Similarly, we have

$$W(u,\bar{u};\xi_i+0)= W(u,\bar{u};\xi_i-0)\ (i=1,2) \qquad (21)$$

By virtue of (20)-(21) the right hand side of (18) is zero. But since $u(x)$ is eigenfunctions, this means that $\lambda-\bar{\lambda}=0$, that is the eigenvalue λ is real.

Theorem 4. If αβ=1, then every eigenvalue has real-valued eigenfunction.

Proof. Let $(\lambda, u(x))$ be any eigenpair of the 3-SLBTP (1)-(3). Let $u(x,\lambda) = v(x,\lambda) + iw(x,\lambda)$, where v and w are real-valued functions.Since $(\bar{\lambda}, \bar{u}(x))$ is also eigenpair and λ **is a** real number it is follows that the function $\bar{u}(x,\lambda) = v(x,\lambda) - iw(x,\lambda)$ is also an eigenfunction corresponding to the same eigenvalue λ. Obviously both real-valued functions

$$v(x) = \frac{u(x)+\bar{u}(x)}{2} \quad \text{and} \quad w(x) = \frac{u(x)-\bar{u}(x)}{2i}$$

satisfy the 3-SLBTP (1)-(3). Obviously at least one of them is not identically zera and, therefore, is also an real-valued eigenfunction corresponding to the same eigenvalue λ. The proof is complete.

Theorem 5. If αβ=1, then the eigenfunctions of a the 3-SLBTP (1)-(3) corresponding to distinct eigenvalues are orthogonal, that is, if (λ_1, u_1) and (λ_2, u_2) are two eigenpairs with $\lambda_1 \neq \lambda_2$, then

$$\sum_{i=1}^{3} \int_{\Omega_i} u_1(x)u_2(x)dx = 0 \qquad (22)$$

Proof. Writing down the equations satisfied by the eigenfunctions $u_1(x)$ and $u_2(x)$ and multiplying the equation for $u_1(x)$ with $u_2(x)$ and vice versa, we have

$$-u_1''(x)\, u_2(x) + q(x)u_1(x)u_2(x) = \lambda_1 u_1(x)\, u_2(x)$$

$$-u_2''(x)\, u_1(x) + q(x)u_2(x)u_1(x) = \lambda_2 u_2(x)\, u_1(x)$$

Taking the difference of these equalities, we get

$$-\frac{d}{dx} W(u_1,u_2;x)=(\lambda_2-\lambda_1)u_1(x)u_2(x) \qquad (23)$$

Integrating the last equality, yields

$$(\lambda_2-\lambda_1)\sum_{i=1}^{3}\int_{\Omega_i} u_1(x)u_2(x) = -W(u_1,u_2;x)|_a^b$$

$$-JW(u_1,u_2;x)|_{\xi_1} - JW(u_1,u_2;x)|_{\xi_2} \qquad (24)$$

Reasoning in exactly the same way as in the proof of Theorem 3, we find that the right hand side of th last equality is equal to zero. Since $\lambda_1 \neq \lambda_2$, we get the desired (22).

Theorem 6. Suppose that αβ=1, then for any eigenpair $(\lambda, u(x))$ of 3-SLBTP (1)-(6) the equality

$$\lambda = \frac{\sum_{i=1}^{3}\int_{\Omega_i}(|u'(x)|^2 + q(x)|u(x)|^2)\,dx}{\sum_{i=1}^{3}\int_{\Omega_i}|u(x)|^2 dx} \qquad (25)$$

holds.

Proof. Appliying the identity (5) we obtain

$$\lambda\sum_{i=1}^{3}\int_{\Omega_i}|u(x)|^2=\sum_{i=1}^{3}\int_{\Omega_i}\bar{u}\,L(u)dx$$

$$-u'(x)v(x)|_a^b - J\left(u'(x)v(x)\right)|_{\xi_1}$$

$$-J\left(u'(x)v(x)\right)|_{\xi_2}$$

$$+\sum_{i=1}^{3}\int_{\Omega_i}(|u'(x)|^2+q(x)|u(x)|^2)dx \qquad (26)$$

Since αβ=1 and u and $\bar{u}$ satisfy the boundary-transmission conditions (2)-(6) then we have

$$u'(b)\bar{u}(b)=u'(a)\bar{u}(b) \qquad (27)$$

and

$$J\left(u'(x)u(x)\right)|_{\xi_1}= J\left(u'(x)u(x)\right)|_{\xi_2} = 0 \qquad (28)$$

Subsitung (27) and (28) into (26) yields the desired (25) .

Corollary 7. Let $(\lambda, u(x))$ be any eigenpair of 3-SLBTP (1)-(6). If αβ=1, then the inequality

$$\lambda \geq \{\inf q(x) | x \in \cup_{i=1}^{3} \Omega_i\}$$

Holds, that is the eigenvalues are bounded from below by the number $\{\inf q(x) | x \in \cup_{i=1}^{3} \Omega_i\}$.

Corollary 8. Let αβ=1. Then the set of eigenvalues of 3-SLBTP (1)-(3) is bounded from below.

Corollary 9. Let αβ=1 and λ_1 be the first eigenvalue. Then the equality

$$\lambda_1 = \inf \frac{\sum_{i=1}^{3} \int_{\Omega_i} (|\mathcal{v}'|^2 + q|\mathcal{v}|^2) dx}{\sum_{i=1}^{3} \int_{\Omega_i} |\mathcal{v}|^2 dx} \qquad (29)$$

is true, where the infumum is taken over all functions

$$\mathcal{v} \in \left(\oplus_{i=1}^{3} C^2(\Omega_i)\right) \cap \left(\oplus_{i=1}^{3} C^1(\overline{\Omega_i})\right)$$

satisfying boundary-impuulsive conditions (2)-(6).

EXAMPLE

Note that the equality (29) can be used to obtain an approximate value of the first eigenvalue using the test functions

$$\mathcal{v} \in \left(\oplus_{i=1}^{3} C^2(\Omega_i)\right) \cap \left(\oplus_{i=1}^{3} C^1(\overline{\Omega_i})\right).$$

Satisfying the boundary-impuulsive conditions (2)-(6). To show this let us consider the following simple special case of the three-interval Sturm-Liouville boundary impulsive problem consisting of the Sturm-Liouville equation

$$u'' + \lambda u = 0 \qquad (30)$$

defined on three non-intersecting intervals [0,2), (2,4) and (4,6], the boundary conditions, given by

$$u(0) = -u(6), \quad u'(0) = -u'(6) \qquad (31)$$

and four additional impulsive conditions, given by

$$u(2-0) = -u(2+0) \qquad (32)$$

$$u'(2-0) = -u'(2+0) \qquad (33)$$

$$u(4-0) = -u(4+0) \qquad (34)$$

$$u'(4-0) = -u'(4+0) \qquad (35)$$

It is easy to verify that the trial function

$$u(x) = \begin{cases} x, & for \quad x\epsilon\,[0,1) \\ -x+2, & for \quad x\epsilon\,[1,2) \\ x-2, & for \quad x\epsilon\,[2,3) \\ -x+4, & for \quad x\epsilon\,[3,4) \\ x-4, & for \quad x\epsilon\,[4,5) \\ -x+6, & for \quad x\epsilon\,[5,6] \end{cases}$$

Satisfies the boundary conditions (31) and the mpulsive conditions (32)-(35).

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