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Perfect Squares and Quadratic Forms

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Abstract – In this note we define the inner product of two vectors by a new form. By this way, we show that every perfect square is a quadratic form of Gram matrix of coefficients of related linear combination. Moreover, we give a different proof that determinant of the quadratic matrix of a perfect square equals to zero. We can obtain an equivalence relation between the quadratic matrices of the same perfect square. It means that our method gives a new aspect of quadratic forms and Pythagorean triples are very useful in the obtaining of the equivalent quadratic matrices of a perfect square.

Keywords – Perfect Squares, Quadratic Forms, Determinant.

I. INTRODUCTION

With respect to Stickelberger's Discriminant Theorem, the discriminant of a number field is congruent to 0 or 1 modulo 4 [1]. A generalization of this theorem showed that there is an equality between the discriminant of a number field and the determinant of the Gram matrix of the traces of basis elements of the related number field [1]. It is understood that the Gram matrix plays a key role in the computation of the discriminant of a number field or determinant of the matrix of the Gram matrix of pairwise traces of basis elements.

The discriminant appears in many mathematical problems for example a quadratic equation $ax^2 + bx + c$ with $a, b, c \in R$ has a real root if and only if the discriminat of the equation is nonnegative.

In this note we define the inner product of two vectors by a new form. By this way, we show that every perfect square is a quadratic form of Gram matrix of coefficients of related linear combination. Moreover, we give a different proof that determinant of the quadratic matrix of a perfect square equals to zero. We can obtain an equivalence relation between the quadratic matrices of the same perfect square. It means that our method gives a new aspect of quadratic forms and Pythagorean triples are very useful in the obtaining of the equivalent quadratic matrices of a perfect square. More information about quadratic forms can be found in the books[2,3].

Let V be a vector space with a fixed basis B. To the basis B, a quadratic form is defined by a symmetric matrix Q

$$q(v) = v^t Q v.$$

Let x be a linear combination of $x_1, x_2, \dots, x_n$ with coefficients $a_1, a_2, \dots, a_n$ and y be a linear combination of $x_1, x_2, \dots, x_n$ with coefficients $b_1, b_2, \dots, b_n$ such that

$$x = \sum_{i=1}^{n} a_i x_i$$

$$y = \sum_{i=1}^{n} b_i x_i$$

Then the inner product of x, y can be defined by

$$\langle x, y \rangle = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

II. RESULTS

Propositon 1. Let x be a linear combination of $x_1, x_2, \dots, x_n$ with coefficients $a_1, a_2, \dots, a_n$ respectively. Then x^2 is a quadratic form of Gram matrix of the coefficients $a_1, a_2, \dots, a_n$.

Proof. Since x is a linear combination of $x_1, x_2, \dots, x_n$ with coefficients $a_1, a_2, \dots, a_n$, we can write that

$$x = \sum_{i=1}^{n} a_i x_i.$$

By this way, the square of x can be written as a quadratic form

$$x^2 = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & & a_na_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = v^t Q v$$

It is understood that Q is the Gram matrix of coefficients of $a_1, a_2, \dots, a_n$. This completes the proof.

If a binary quadratic form $f(x,y) = ax^2 + bxy + cy^2$ $(a, b, c \in R)$ is perfect square, then discriminant $d = b^2 - 4ac$ of $f(x,y)$ equals to zero. If $f(x,y)$ is witten as $f(x,y) = v^t Q v$ and a perfect square, then the determinant of the quadratic matrix $Q = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ equals to zero.

By Proposition 1, we can give an alternative proof that the determinant of the quadratic matrix of a perfect square equals to zero.

Propositon 2. If a phrase is perfect square, then the determinant of quadratic matrix of this phrase equals to zero.

Proof. By Proposition 1, if x is a linear combination of $x_1, x_2, \dots, x_n$ with coefficients $a_1, a_2, \dots, a_n$, we can write that

$$x^2 = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & & a_na_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

with the quadratic matrix

$$Q = \begin{bmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & & a_na_n \end{bmatrix}.$$

The determinant of Q equals to

$$\det Q = \begin{vmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & & a_na_n \end{vmatrix}$$

$$= a_1 a_2 \dots a_n \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & & a_n \end{vmatrix}$$

$$= 0.$$

The result of Proposition 1 can be extended to integers. Even though the set of integers Z is not a vector space, the square of an integer n can be written as a quadratic form of the Gram matrix of its digits as follows. Assume that $n = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0$. The square of n is expressed by

$$n^2 = v^t Q v$$

$$= \begin{bmatrix} 10^n & 10^{n-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_n a_n & a_n a_{n-1} & \cdots & a_n a_0 \\ a_{n-1} a_n & a_{n-1} a_{n-1} & \cdots & a_{n-1} a_0 \\ \vdots & \vdots & \ddots & \vdots \\ a_0 a_n & a_0 a_{n-1} & & a_0 a_0 \end{bmatrix} \begin{bmatrix} 10^n \\ 10^{n-1} \\ \vdots \\ 1 \end{bmatrix}$$

such that Q is the Gram matrix of digits of n.

Theorem 3. Let n be integer. Then n^2 is a quadratic form of different quadratic matrices with the $[10^n \quad 10^{n-1} \quad \cdots \quad 1]$ and $[10^n \quad 10^{n-1} \quad \cdots \quad 1]\,'$. This is an equivalence relation between the quadratic matrices.

Proof. The proof is clear and we omit the details.

In order to exemplify Theorem 3, we give an example by Pythagorean Triples. (7,24,25) and (15,20,25) are two Pythagorean Triples. By this way we can find three quadratic matrix of 25. The first one is obtained from gram matrix of its digits. The second one and the third one can be obtained from the sum of the Gram matrices of the other components as follows.

$$25^2 = \begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$25^2 = 7^2 + 24^2$$

$$= \begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 49 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix} + \begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 8 & 65 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$25^2 = 15^2 + 20^2$$

$$= \begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 5 & 25 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix} + \begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 1 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ 5 & 25 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

Then we can say that the following quadratic matrices are equivalent.

$$\begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix}, \begin{bmatrix} 4 & 8 \\ 8 & 65 \end{bmatrix}, \begin{bmatrix} 5 & 5 \\ 5 & 25 \end{bmatrix}$$

Theorem 4. Let b be an odd integer as b^2 equals to difference of squares of two consecutive numbers $n = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0$ and $n+1$. Then, the qudratic matrix of b^2 can be reduced to a matrix whose components are given for non zero entries $a_{1,n+1} = a_n,\ a_{2,n+1} = a_{n-1}, \dots, a_{n+1,n+1} = 2 \times a_0 + 1$ with symmetric entries.

Proof. Assume that b be an odd integer. Then b is one of a Pythagorean Triple as the other components are consecutive integers $n = (b^2 - 1)/2$ and $n + 1 = (b^2 + 1)/2$. By this way the quadratic matrix of b^2 equals to difference of quadratic matrices of $(n+1)^2$ and n^2.

If $n = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0$, then the difference of qudratic matrices of $(n+1)^2$ and n^2 equals to quadratic matrix with non zero entries $a_{1,n+1} = a_n,\ a_{2,n+1} = a_{n-1}, \dots, a_{n+1,n+1} = 2 \times a_0 + 1$ with symmetric entries.

Example 5. $(569, 161880, 161881)$ is a Pythagorean triple. Then the quadratic matrix of 569^2 is obtained as in the following matrix by Thorem 4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 8 \\ 1 & 6 & 1 & 8 & 8 & 1 \end{bmatrix}$$

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