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Discrete fractional numerical analysis on the shallow water wave theory

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Abstract – In order to address the wave height distribution in any region, from deep ocean to shallow water, coastal engineering, electromagnetic wave propagation and scattering, and acoustics, this study introduces numerical methods to tackle a variety of problems in wave theory. At that point, numerical analysis and techniques become useful in assisting us in obtaining the most accurate approximation possible for our barrier.

The core formulas of our numerical analysis technique are the linearized wave equations with unknown functions only at the water surface, like the particle velocity components and the elevation of the water surface, which are derived from the Eulerian equations of motion and continuity assuming small amplitude in constant water depth.

We quantify the accuracy of discretization solution techniques such as finite difference or finite elements schemes in powers of a discretization step size h. The Nemerov system is useful. A method with p > 2 is typically referred to as a higher order method, and one with error O (h^p) is said to be of order p.

Keywords – shallow waves, approximations, waves distribution, coastal engineering, mathematical techniques, errors.

I. Introduction

The ability of mathematics to provide solutions to idealized problems and to provide explicit solution forms to less-idealized problems has provided a huge corpus of knowledge which can be used in coastal and ocean engineering as well as electromagnetism. Numerical methods are powerful tools that enable the solution of the differential, integral, or algebraic equations that arise in engineering and physics problems. Hydraulic model testing is typically how a realistic wave height distribution in a harbour is obtained. It would be very helpful to examine an ideal breakwater arrangement if we could theoretically treat all harbour disturbance effects, such as wave diffraction, reflection, and refraction. This paper provides instances of wave height distribution calculations in arbitrary-shaped harbours with fixed or variable water depths.

Propagation of small amplitude waves in the region of constant water depth of ideal fluid is treated in this analysis and irrotational motion is assumed. We outline computational approaches that are practical for addressing issues in coastal and ocean engineering that often don't need for significant analytical approximations and can produce highly accurate findings. Most recently, Berkhoff (Berkhoff, 2002) has discussed the computation of combined refraction-diffraction. These mathematical techniques are all used to address boundary value problems involving fundamental wave equations. However, the authors have investigated how to determine the height and flow distribution of long waves in a harbour of any shape from the perspective of the effectiveness of breakwaters as a tsunami defence.

Our approach of numerical analysis may be used to determine the wave height distribution around any arrangement of breakwaters, and it is simple to add additional vertical walls behind the breakwaters. Differential equations are produced by modelling the dynamics of motion mathematically. It is generally impossible to find the analytical solution functions for these equations, with the exception of a small number of special cases. As a result, in order to describe the functions, we must study asymptotic and approximate solutions as well as run deterministic or stochastic computer simulations.

But in order to perform numerical computations, we need to replace continuous problems with discrete ones, which makes it unavoidable to introduce errors.

Often when dealing with real life problems such as the estimation of oceans currents or electromagnetic waves it is nearly impossible to calculate the exact solution.

II. Materials and Method

2.1 Equations of motion and of continuity and boundary conditions

Most applications in coastal and ocean engineering are of such a large physical extent that detailed solution of the primitive equations of fluid mechanics throughout the flow field has not been feasible. However, in most methods used two assumptions are made which enable the problem to be reduced to that of solving Laplace's equation. In this case several methods exist which solve the whole flow field by summary methods, such as recognizing that the solution depends on values only around the boundary, or by eigenfunction expansions. Propagation of small amplitude waves in the region of constant water depth of ideal fluid is treated in this analysis and irrotational motion is assumed.

The Eulerian equations of motion and of continuity and boundary conditions at the water surface and the bottom for a linearized wave are as follows:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (1.1)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (1.1)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad (1.1)$$

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} + \frac{\partial w}{\partial t} = 0 \quad (1.2)$$

$$w = \frac{\partial n}{\partial t}, \text{ at } z = n \quad (1.4)$$

$$p = 0, \text{ at } z = n \text{ and } w = 0, \text{ at } z = -h$$

where u, v, w are components of water particle velocity, p is pressure, n is water surface elevation, and h is water depth.

It might be thought that this is a remarkable result, that the Euler equations, the set of three nonlinear dynamic equations of fluid dynamics can be reduced to a single partial differential equation, linear in the dependent variable φ as shown down below.

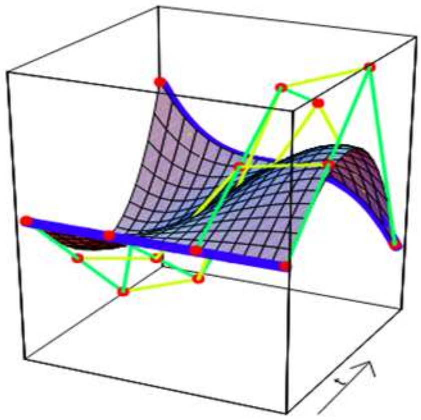

Fig. 1. A solution of the 1D-wave equation as a two-dimensional surface.

In this case, there exists a velocity potential φ (Batchelor, 1997 #2.7) such that the velocity vector u is given by the gradient $u = \nabla\varphi$, which identically satisfies the condition for irrotationality, $\nabla \times u = 0$.

In cartesian co-ordinates (x, y, z) the gradient operator is:

$$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

which gives the velocity components:

$$u = \partial\varphi/\partial x,\ v = \partial\varphi/\partial y, \text{ and } w = \partial\varphi/\partial z$$

Throughout this work we will choose the co-ordinates in the horizontal plane to be x and z, with y the vertical co-ordinate. We also assume that the fluid is incompressible, in which case mass conservation is satisfied by the equation $\nabla.u = 0$, so that substituting the condition for irrotationality, we obtain:

$$\nabla^2\varphi = 0 \tag{1.5}$$

showing that the velocity potential must satisfy Laplace's equation throughout the flow domain. In

cartesian co-ordinates this becomes:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{1.6}$$

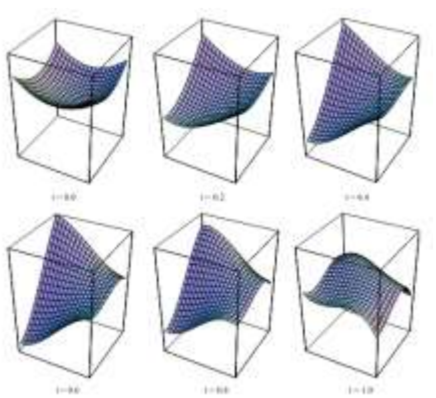

Fig. 2. Idealized wave diffraction. Six states of the evolution of a solution for the 2D-wave equation with border control points

2.2 *Derivation of basic wave equations at water surface with kinematic boundary conditions*

B´ezier solutions of the 2D-wave equation If a thin elastic membrane of uniform areal density is stretched to a uniform tension and if, in the equilibrium position, the membrane coincides with the xy-plane, then the small transverse vibration u(x, y, t) of the point (x, y) of the membrane satisfies an equation in which the two dimensional Laplacian of u is proportional to the second partial derivative with respect to t,

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u = \Delta u = c^2 \frac{\partial^2 u}{\partial t^2}.$$

This equation is called the two-dimensional wave equation.

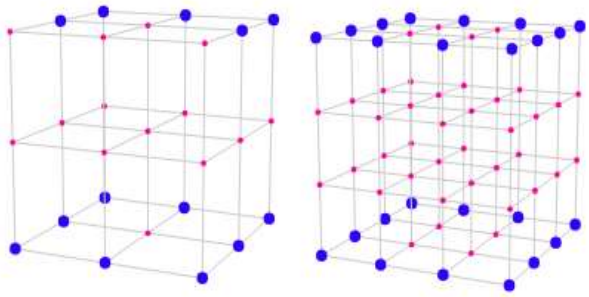

Fig. 3. Bezier curves Left n = 2. Right n = 3

- t is any parameter where 0 <= t <= 1
- P(t) = Any point lying on the bezier curve
- B_i = i^{th} control point of the bezier curve
- n = degree of the curve
- $J_{n,i}(t)$ = Blending function

$C(n,i)t^i(1-t)^{n-i}$ where C(n,i) = n! / i!(n-i)! 2.1

The de Casteljau's algorithm recursively evaluates the control points to determine the position of points on the Bezier curve. This algorithm provides flexibility in shape control and is extensively used in shape design, animation, and modeling [2].

The curve is completely contained in the convex hull of its control points.
So, the points can be graphically displayed & used to manipulate the curve intuitively [7].

2.3 B-Spline Techniques for spline curves:

Given n + 1 control points P_0, P_1, ..., Pn and a knot vector U = { u0, u1, ..., um }, the B-spline curve of degree p defined by these control points and knot vector U is

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i \quad 2.2$$

where Ni,p(u)'s are B-spline basis functions of degree p. The form of a B-spline curve is very similar to that of a Bezier curve. Unlike a Bezier curve, a B-spline curve involves more information, namely: a set of n+1 control points, a knot vector [17, 18, 19, 20] of m+1 knots, and a degree p. Note that n, m and p must satisfy m = n + p + 1. More precisely, if we want to define a B-spline curve of degree *p* with *n + 1* control points, we have to supply *n + p + 2* knots *u0, u1, ..., un+p+1*. The degree of a B-spline basis function is an input, while the degree of a Bezier basis function depends on the number of control points [25]. To change the shape of a B-spline curve, one can modify one or more of these control parameters: the positions of control points, the positions of knots, and the degree of the curve. If the knot vector does not have any particular structure, the generated curve will not touch the first and last legs of the control polyline it is called *OPEN*.

We may want to clamp the curve so that it is tangent to the first and the last legs at the first and last control points, respectively, as a Bezier curve does. To do so, the first knot and the last knot must be of multiplicity p+1. This is called *CLAMPED*, [2] [8].

If we repeat the process we get a *CLOSED B-SPLINE*.

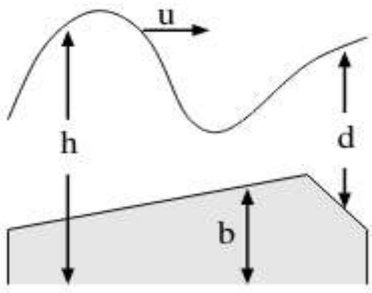

Fig. 4. One-dimensional shallow water wave

III. RESULTS SURFACE FITTING TECHNIQUES

3.1 *Surface Subdivision*

Surface subdivision is a surface fitting technique that iteratively refines a mesh representation, resulting in smoother and more detailed surfaces. The basic idea behind surface subdivision is to divide each polygonal face into smaller sub-faces and adjust the positions of the vertices based on certain rules or algorithms. This process is repeated iteratively until the desired level of smoothness and detail is achieved [5].

Surface subdivision techniques, such as the Loop subdivision algorithm or the Catmull-Clark subdivision algorithm, are commonly used in 3D modeling, sculpting, terrain modeling, and medical imaging. These techniques provide precise control over the shape and detail of surfaces, enabling the creation of realistic and visually appealing 3D objects [3]. Quadrilateral based meshes generally use Catmull-Clark, while triangular based meshes generally use loop subdivision.

3.2 *Surface Loop subdivision:*

Loop subdivision, do not involve explicit equations but operate based on vertex and face subdivision rules, The algorithm begins by calculating new positions for each vertex in the mesh based on a weighted average of its neighboring vertices. These weights are determined by the topology of the mesh and ensure that the new positions preserve the overall shape of the surface [5]. After updating the vertex positions, the algorithm creates new edges and faces by connecting the newly generated vertices. This step increases the level of detail in the mesh and improves the smoothness of the surface. The loop subdivision algorithm is recursive, meaning that it can be applied multiple times to achieve even higher levels of refinement. Each iteration further smooths the surface and adds more detail to the mesh. Loop subdivision has several advantages. It produces visually pleasing surfaces with smooth and continuous curvature. It also maintains the overall shape of the original mesh while adding additional geometric detail. Additionally, it is relatively efficient in terms of computational complexity compared to other subdivision techniques [3].

Catmull-Clark is an algorithm that maps from a surface (described as a set of points and a set of polygons with vertices at those points) to another more refined surface. The resulting surface will always consist of a mesh of quadrilaterals.
The process for computing the new locations of the points works as follows when the surface is free of holes. For each face, a face point is created which is the average of all the points of the face. For each edge, an edge point is created which is the average between the center of the edge and the center of the segment made with the face points of the two adjacent faces. For each vertex point, its coordinates are updated [10].

3.3 Interpolation

Interpolation is another surface fitting technique used in computer graphics. It involves fitting a smooth surface through given control points or samples. The goal is to find a surface that passes through the given points while maintaining smoothness and continuity. Various interpolation methods can be employed, such as polynomial interpolation, spline interpolation, or radial basis function interpolation. These techniques are used in applications like 3D modeling, image reconstruction, and medical imaging to create smooth and continuous surfaces based on available data points [3]. 3D transformations are fundamental operations in computer graphics that allow for the manipulation and positioning of objects in 3D space. The three main types of 3D transformations are translation, rotation, and scaling.

Translation involves shifting objects along the x, y, and z axes. Rotation changes the orientation of objects around a specified axis or point. Scaling adjusts the size of objects, either uniformly or along specific axes

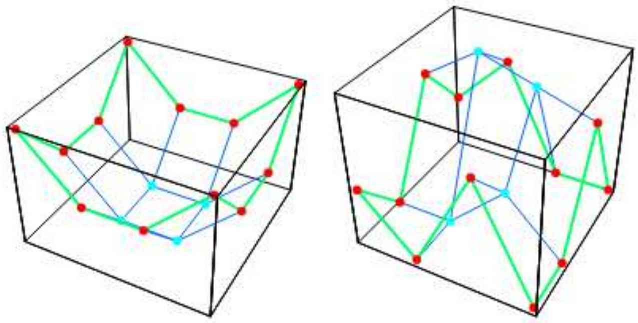

Fig. 5. Initial state Final state. Visual quality of computer-generated scenes

These transformations are represented using transformation matrices and are crucial for object manipulation, camera movement, and scene composition in computer graphics. By applying appropriate transformation matrices, practitioners can achieve dynamic and interactive 3D graphics, enhancing the realism and visual quality of computer-generated scenes [4].

3.4 Simulations

A study of Eqs. 8 for $t = m = n = 3$ shows that given the border control points of the initial and final states (2×12) of the Bezier membrane, the other control points ($4 + 16 + 16 + 4$) of the whole control web are determined. In the next Figure the independent border control points of the initial and final states are plotted thicker.

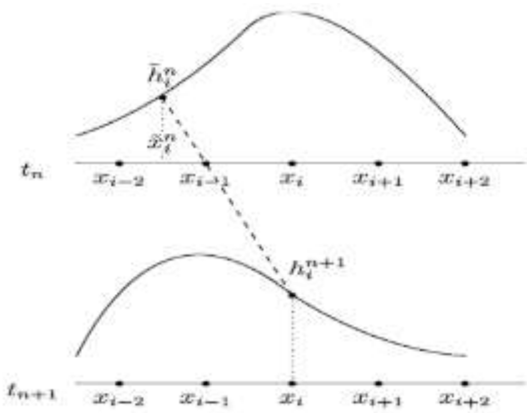

Fig. 6. With the semi-Lagrangian scheme, the Lagrangian derivative is approximated along particle trajectories

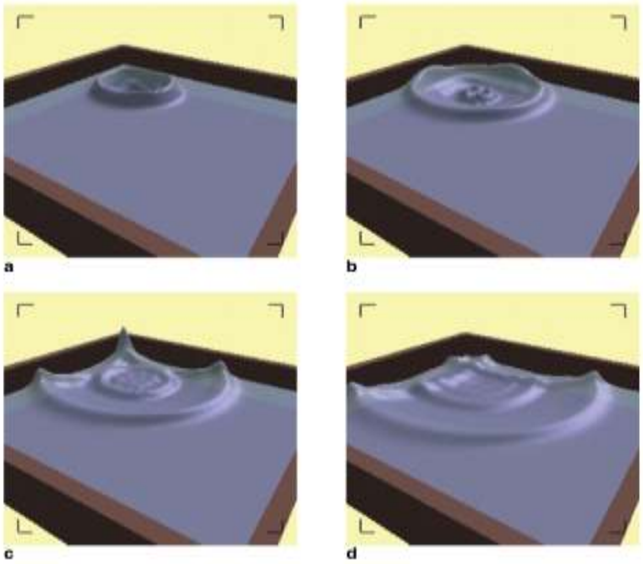

Fig. 7. Simulation of water waves in a rectangular pool. a First frame; b second frame; c third frame; d fourth frame

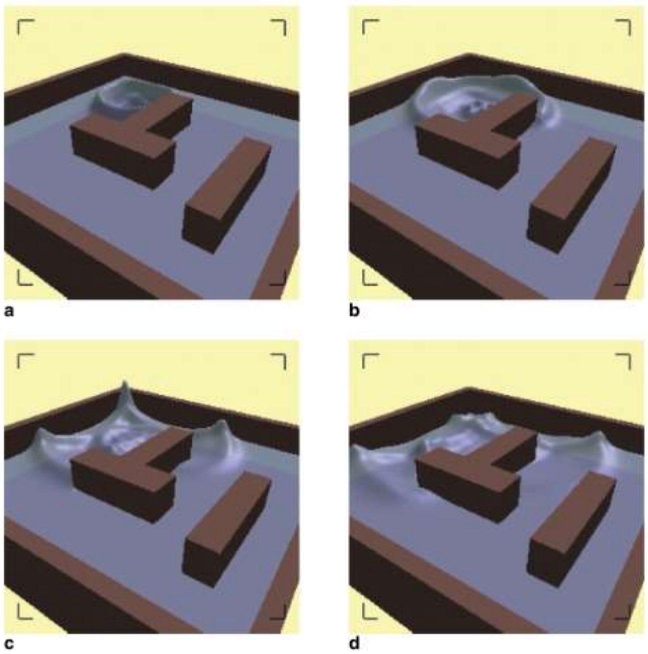

Fig. 8. Beizer interpolation scene 2

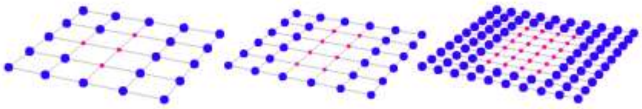

Fig. 9. Representation for n = 4 (left), n = 5 (center) and n = 9 (right) of the independent control points. 3

In order to validate the effectiveness and practicality of the numerical methods discussed, simulations were conducted using software tools such Python and online compiler Replit. These simulations involved generating and manipulating various graphical elements.

IV. DISCUSSION

Based on the analysis of Mean Squared Error (MSE) and Peak Signal-to-Noise Ratio (PSNR) metrics for image resizing using different interpolation methods, we can draw the following conclusions:

4.1 Resizing Down: When comparing the resized images to the original ones after downsizing, the interpolation methods exhibit distinct characteristics:

Nearest Neighbor interpolation performs the poorest, as it introduces noticeable differences and reduces the similarity between the resized and original images.

Linear interpolation demonstrates the most significant change in similarity, resulting in resized images that are more similar to the original images compared to other methods. This indicates that linear interpolation tends to preserve the overall structure and visual fidelity during the downsizing process.

Cubic and Lanczos interpolation methods show similar performance, as they produce resized images that are relatively close to the original images. However, considering their computational complexity, the similarity improvement achieved by these methods may not justify the additional resources required.

4.2 Resizing Up

When comparing the resized images to the original ones after upsizing, similar trends emerge as observed during the downsizing process:

Nearest Neighbor interpolation still demonstrates a lower level of similarity compared to the original images, although the difference is not as pronounced as during downsizing. This method tends to introduce blocky artifacts and limited smoothness during the upsizing process.

Linear interpolation continues to exhibit a significant improvement in similarity, making the resized images more similar to the originals. It maintains the overall structure and visual fidelity, enhancing the upsized images' quality.

V. CONCLUSION

In conclusion, based on both MSE and PSNR metrics, linear interpolation consistently performs well in terms of maintaining similarity between the resized and original images, both when downsizing and upsizing. Nearest Neighbor interpolation shows poorer results, while Cubic and Lanczos interpolation methods offer relatively good performance but may not justify their increased computational complexity. It is important to consider the specific requirements of the image resizing task and the trade-off between similarity and computational efficiency when selecting an interpolation method for image processing.

Alternatives to affirmative action, such as socioeconomic-based admissions or hiring practices, are being touted as potentially more fair methods of achieving diversity without overtly considering race or gender. Such alternatives, however, also come with their own set of challenges and implications, necessitating in-depth research and pilot programs to assess their viability.

Although some may argue that affirmative action can lead to reverse discrimination, the evidence suggests that it is an important and effective instrument for achieving greater equality. Affirmative action policies may not always be enough to help corporations and colleges reach their diversification and inclusion objectives.

In summarizing affirmative action, we reflect on a complex landscape characterized by divergent views and constant evolution. Through this blog post, we have explored the historical roots, the myriad arguments, and the real-world applications of affirmative action. Whether these policies stand as a remedy to past discrimination or a challenge to meritocratic principles depends largely on where one stands in the ongoing debate.

Cubic and Lanczos interpolation methods show similar performance, as they produce resized images that are relatively close to the original images. However, considering their computational complexity, the similarity improvement achieved by these methods may not justify the additional resources required. The discourse extends an invitation to engage in constructive dialogue, deepen understanding, and potentially influence the course of affirmative action in the future. The pursuit of true workplace diversity is far from a finished journey, but through continued evaluation and thoughtful conversation, progress can be made.

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