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Caputo Fractional Operator on shallow water wave theory

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Abstract – In this section, we aim to introduce fractional operators that are applicable to real-world scenarios, featuring both singular and nonsingular kernels. Given the extensive scope of fractional calculus, which encompasses numerous mathematical and practical domains, we employ precise definitions alongside shallow water wave theory models. This study offers numerical solutions for various wave theory challenges, including wave height distribution across different depths, coastal engineering, and electromagnetic wave phenomena. Numerical analysis and strategic approaches are crucial for obtaining highly accurate estimates for barriers. The core of our numerical method lies in linearized wave equations, focusing on variables such as particle velocity and surface elevation, derived from Eulerian motion and continuity equations under the assumption of small amplitude in consistent water depth. We evaluate the precision of discretization methods (e.g., finite difference or finite element techniques) based on step size (h), noting that approaches where $p > 2$ often present higher error rates and are considered advanced.

Keywords – Fractional operator, singular kernel, numerical simulation, water wave, acoustics

I. Introduction

Mathematics is essential in addressing both straightforward and complex challenges, playing a significant role in areas such as coastal and ocean engineering, as well as in electromagnetism. Numerical methods are crucial for solving fractional order differential, integral, and algebraic equations that frequently arise in engineering and physics. This work aims to introduce the definitions of fractional calculus and demonstrate its application in engineering models. Typically, hydraulic model testing is employed to generate accurate wave height distributions in harbors. However, the ability to theoretically analyze all harbor disturbances, including wave diffraction, reflection, and refraction, would greatly enhance the study of optimal breakwater designs. This paper presents examples of wave height distribution calculations in harbors with various geometries, considering both constant and variable water depths. The analysis primarily focuses on small amplitude wave propagation in regions with constant water depth, assuming irrotational motion. Practical computational methods are discussed, which address challenges in coastal and ocean engineering without heavily relying on analytical approximations, leading to highly accurate results. Recently, Berkhoff (2022) examined the computation of combined refraction-diffraction effects, which involve mathematical approaches to boundary value problems and the application of fractional differential operators with both singular and nonsingular kernels. Additionally, we evaluate the effectiveness of breakwaters in mitigating tsunami impacts by analyzing wave height and flow distribution in harbors of various configurations. Our numerical analysis and simulations, conducted using MATLAB, provide a flexible method for determining wave height distribution around different breakwater configurations, with the option to add vertical walls behind them. The motion dynamics are modeled mathematically through differential equations, which are typically unsolvable analytically, except in special cases. Thus, we explore asymptotic and approximate solutions and perform deterministic or stochastic computer simulations. However, translating continuous problems into discrete ones for numerical computation introduces errors. In practical applications, such as predicting ocean currents or electromagnetic waves, obtaining an exact solution is often nearly impossible. Given this context, fractional calculus will be applied to model shallow water waves, and numerical results will be generated. However, we first need to cover some basic definitions related to fractional calculus.

II. HISTORY OF FRACTIONAL CALCULUS

Applied mathematics is vital in academia, with scientists frequently utilizing it to solve practical issues. This extensive domain includes areas such as differential equations, fractional differential equations, mathematical modeling, calculus on time scales, and dynamic systems. Fractional calculus, a significant branch, has a rich history marked by contributions from numerous mathematicians. The development of integer-order calculus in the 17th century by Sir Isaac Newton and Gottfried Wilhelm Leibniz laid the groundwork for later advances. Newton's contributions, in particular, had a profound impact on the evolution of calculus. As the field progressed, fractional-order derivatives and integrals began to receive considerable attention. Without Newton's foundational work, fractional calculus and related mathematical areas might not have achieved their current level of significance. Today, advancements in mathematics and technology have expanded the application of applied mathematics across a broad range of scientific fields. [1]-[2]-[3]-[4]-[5]-[6]-[7]-[8]-[9]-[10]-[11]-[12]-[13]-[14]-[15]-[16]-[17]-[18]-[19]-[20]-[21]-[22]-[23]-[25]-[26]-[27]-[28]-[29]-[30]-[31]-[32]-[33]-[34]-[35]-[36]-[37]-[38]-[39]-[40]-[41]-[42]-[43]-[44]-[45]-[46]-[47]-[48]-[49]-[50]-[51]-[52]-[53]-[54]-[55]-[56]-[57]-[58]-[59]-[60]-[61]-[62]-[63]-[64]. This has led to significant research in fractional calculus, both in theoretical and applied settings. Modern fractional calculus encompasses various areas, including discrete, local, and non-local fractional calculus, as well as conformable and deformable fractional calculus in continuous scenarios. Research has extensively examined fractional operators with both singular and non-singular kernels. Additionally, Abdeljawad and Baleanu introduced a novel non-local fractional derivative featuring a Mittag-Leffler non-singular kernel and integration by parts, contributing to the advancement of the field.in [65]. Jarad, F Abdeljawad, T and Baleanu, D also utilized Caputo's modification in their work on generalized fractional derivatives. [66].

Additionally, Baleanu, D, Hasanbadi, M., Vaziri, A.M. and Jajarmi, A. developed a new strategy for addressing HIV/AIDS transmission using general fractional modeling combined with an optimal control approach, [67]. Peter, O.J., Shaikh, A.S., Ibrahim, M.O., Nisar, K.S., Baleanu, D., Khan, L., and Abioye, A.I., used the Atangana-Baleanu fractional operator to analyze and model the dynamics of the Covid-19 outbreak in Nigeria. [68]. Lastly, fractional analysis was applied to models that include HIV with CD++ T-cells and the SIRS-SI disease model, [69]-[70].

III. DEFINITIONS

3.1 Grunwald-Letnikov fractional derivatives

In [71], an approach is explored that aims to unify two concepts traditionally treated separately in calculus: the derivative of integer order ρ and ρ-fold integrals. This approach originates from the definition of limits in calculus.Let, $y = p(t)$ is a continuous function then the first order derivative,

$$\mathfrak{Y}p(t) = \lim_{h\to 0}\frac{p(t+h)-p(t)}{h}$$

$$\mathfrak{Y}^2p(t) = \mathfrak{Y}(\mathfrak{Y}p(t)) = \mathfrak{Y}\left\{\lim_{h\to 0}\frac{p(t+h)-p(t)}{h}\right\}$$

$$= \lim_{h\to 0}\frac{\mathfrak{Y}\{p(t+h)-p(t)\}}{h}$$
$$= \lim_{h\to 0}\frac{\mathfrak{Y}p(t+h)-\mathfrak{Y}p(t)}{h}$$
$$= \lim_{h\to 0}\left\{\frac{\left(\lim_{h\to 0}\frac{p(t+2h)-p(t+h)}{h}\right)-\left(\lim_{h\to 0}\frac{p(t+h)-p(t)}{h}\right)}{h}\right\}$$
$$\mathfrak{Y}^2 p(t) = \lim_{h\to 0}\frac{p(t+2h)-p(t+h)-p(t+h)+p(t)}{h^2}$$
$$= \lim_{h\to 0}\frac{p(t+2h)-2p(t+h)+p(t)}{h^2}$$
$$\mathfrak{Y}^3 p(t) = \mathfrak{Y}(\mathfrak{Y}p(t)) = \mathfrak{Y}\left\{\lim_{h\to 0}\frac{p(t+2h)-2p(t+h)+p(t)}{h^2}\right\}$$
$$= \lim_{h\to 0}\frac{\mathfrak{Y}\{p(t+2h)-2p(t+h)+p(t)\}}{h^2}$$
$$= \lim_{h\to 0}\frac{\mathfrak{Y}p(t+2h)-2\mathfrak{Y}p(t+h)+\mathfrak{Y}p(t)}{h^2}$$
$$= \lim_{h\to 0}\left\{\frac{\left(\lim_{h\to 0}\frac{p(t+3h)-p(t+2h)}{h}\right)-\left(\lim_{h\to 0}\frac{2p(t+2h)-2p(t+h)}{h}\right)+\left(\lim_{h\to 0}\frac{p(t+h)-p(t)}{h}\right)}{h^2}\right\}$$
$$= \lim_{h\to 0}\frac{p(t+3h)-3p(t+2h)+3p(t+h)-p(t)}{h^3}$$

According to the mathematical induction (see [71]);

$$\mathfrak{y}^k p(t) = \lim_{h\to 0}\frac{1}{h^k}\sum_{r=0}^{k}(-1)^r\binom{k}{r}p(t-rh),\quad k\in\mathbb{Z},$$

where,

$$\binom{k}{r} = \frac{k(k-1)(k-2)\ldots(k-r+1)}{r!} \tag{3.1}$$

and so,

$$\mathfrak{y}^k p(t) = \lim_{h\to 0}\frac{1}{h^k}\sum_{r=0}^{k}(-1)^r\frac{k(k-1)(k-2)\ldots(k-r+1)}{r!}p(t-rh), \tag{3.2}$$

In [71], if we assume k is negative values then;

$$\begin{bmatrix}k\\r\end{bmatrix} = \frac{k(k+1)\ldots(k+r-1)}{r!}$$

So, we have

$$\binom{-k}{r} = \frac{-k(-k-1)\ldots(-k-r+1)}{r!} = (-1)^r\begin{bmatrix}k\\r\end{bmatrix}$$
$$\mathfrak{y}^{-k}p(t) = \lim_{h\to 0}\frac{1}{h^k}\sum_{r=0}^{k}(-1)^r\begin{bmatrix}k\\r\end{bmatrix}p(t-rh),\quad k>0.$$

Theorem 3.1: $\underbrace{\lim_{h\to 0} p_h^{(-k)}(t)}_{nh=t-a} = {}_a\mathfrak{Y}_t^{-k}p(t).$

Proof 3.1: According to the induction theorem, for $k = 1$ we have (see [71]),

$$p_h^{-1}(t) = h\sum_{r=0}^{k} p(t-rh) \tag{3.3}$$

In here $p(t)$ is considered is to be continuous function, we show that (see [71]);

$$\lim_{h\to 0} p_h^{-1}(t) = {}_a\mathfrak{Y}_t^{-1}p(t) = \int_0^{t-a} p(t-s)ds = \int_a^t p(\tau)d\tau \tag{3.4}$$

For $k = 2$ then;

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{2.3 \ldots (2+r-1)}{r!} = \frac{2.3 \ldots (1+r)}{r!} = \frac{2.3 \ldots r(r+1)}{r!} = r+1$$

$$p_h^{(-2)}(t) = h\sum_{r=0}^{k} (r+1)hp(t-rh) \tag{3.5}$$

$$p_h^{(-2)}(t) = h\sum_{r=1}^{k+1} rhp(t-rh) \tag{3.6}$$

For $k = 3$ then;

$$\begin{bmatrix} 3 \\ r \end{bmatrix} = \frac{3.4.5 \ldots (3+r-1)}{r!} = \frac{3.4.5 \ldots (2+r)}{r!} = \frac{2.3.4.5 \ldots r(r+1)(r+2)}{2r!} = \frac{(r+1)(r+2)}{2}$$

$$p_h^{(-3)}(t) = \sum_{r=2}^{k+2} \frac{(r+1)(r+2)}{h} hp(t-rh) \tag{3.7}$$

$$p_h^{(-3)}(t) = \sum_{r=2}^{k+2} (r+1)(r+2)p(t-rh) \tag{3.8}$$

Integrating the relationship (see [71]);

$$\frac{d}{dt}\left({}_a\mathfrak{Y}_t^{-k}p(t)\right) = \frac{1}{(k-2)!}\int_a^t (t-\tau)^{k-2}p(\tau)d\tau = {}_a\mathfrak{Y}_t^{-k+1}p(t).$$

From a to t we satisfy (see [71]);

$$\begin{aligned} {}_a\mathfrak{Y}_t^{-k}p(t) &= \int_a^t \left({}_a\mathfrak{Y}_t^{-k+1}p(t)\right)dt \\ &= {}_aI_t\left({}_a\mathfrak{Y}_t^{-k+1}p(t)\right) \\ &= {}_aI_t\left({}_a\mathfrak{Y}_t^{-k}\, {}_a\mathfrak{Y}_t p(t)\right) = {}_aI_t\, {}_a\mathfrak{Y}_t\left({}_a\mathfrak{Y}_t^{-k}\, p(t)\right) = {}_a\mathfrak{Y}_t^{-k}\, p(t). \end{aligned}$$

$$\begin{aligned} {}_a\mathfrak{Y}_t^{-k+1}p(t) &= \int_a^t \left({}_a\mathfrak{Y}_t^{-k+2}p(t)\right)dt \\ &= {}_aI_t\left({}_a\mathfrak{Y}_t^{-k+1+1}p(t)\right) \\ &= {}_aI_t\left({}_a\mathfrak{Y}_t^{k+1}\, {}_a\mathfrak{Y}_t p(t)\right) \\ &= {}_aI_t\, {}_a\mathfrak{Y}_t\, {}_a\mathfrak{Y}_t^{k+1}\, p(t) \\ &= {}_a\mathfrak{Y}_t^{k+1}\, p(t). \end{aligned}$$

and so,

$$
\begin{aligned}
{}_a\mathfrak{D}_t^{-k}p(t) &= \int_a^t dt \int_a^t \left({}_a\mathfrak{D}_t^{-k+2}\, p(t)\right)dt \\
&= \int_a^t dt \int_a^t dt \int_a^t \left({}_a\mathfrak{D}_t^{-k+3}\, p(t)\right)dt \\
&= \underbrace{\int_a^t dt \int_a^t dt \int_a^t dt \int_a^t dt \ldots \int_a^t dt}_{k-\text{times}}
\end{aligned}
$$

3.2 Riemann-Liouville fractional operator

Integro differential expression of Grunwald-Letnikov fractional operator is identified by (see [71]);

$$
{}_a\mathfrak{D}_t^{k}\, p(t) = \left(\frac{d}{dt}\right)^{n+1} \int_a^t (t-\tau)^{n-k} p(\tau)d\tau, \qquad (n \le k < n+1) \qquad (3.9)
$$

Expression (3.9) represents the widely recognized Riemann-Liouville definition of the fractional derivative. It can be shown that the Grunwald-Letnikov fractional operator, applicable to fractional order derivatives when the function $p(t)$ is $k+1$ times continuously differentiable, can also be derived from expression (3.9) by repeatedly applying integration by parts and differentiation, as demonstrated in [71].

$$
\begin{aligned}
{}_a\mathfrak{D}_t^{k}\, p(t) = \left(\frac{d}{dt}\right)^{n+1} \int_a^t (t-\tau)^{n-k} p(\tau)d\tau &= \sum_{r=0}^{n} \frac{p^{(r)}(a)(t-a)^{-k+r}}{\Gamma(-k+r+1)} + \int_a^t (t-\tau)^{n-k} p^{(n+1)}(\tau)d\tau, \\
&= {}_a\mathfrak{D}_t^{k}\, p(t), \quad (n \le k < n+1), \qquad (3.10)
\end{aligned}
$$

3.3 Caputo fractional operator

The Caputo fractional derivative and integral have significantly advanced fractional calculus, expanding its use in diverse scientific disciplines (see [71]). This operator is particularly useful in domains like engineering, finance, seismology, biological modeling, data analytics, electromagnetic theory, physical simulations, and chemistry.. In [71], Moreover, the Riemann-Liouville approach often results in initial conditions dependent on the limit values of the Riemann-Liouville fractional derivatives at the lower bound $t = a$.

$$
\begin{aligned}
&\lim_{t\to a} {}_a\mathfrak{D}_t^{\beta-1} p(t) = c_1, \\
&\lim_{t\to a} {}_a\mathfrak{D}_t^{\beta-2} p(t) = c_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3.11)\\
&\lim_{t\to a} {}_a\mathfrak{D}_t^{\beta-m} p(t) = c_k,
\end{aligned}
$$

where, c_k, $k = 1,2, \ldots, m$ are given constants. Some definitions and certain solutions were given place [72]-[73]-[74]-[75]. Caputo's definition is defined by (see [71]);

$$
{}_a^C\mathfrak{D}_t^{\beta} p(t) = \frac{1}{\Gamma(\beta-k)} \int_a^t \frac{p^{(k)}(\tau)d\tau}{(t-\tau)^{\beta-k+1}}, \qquad (k-1 < \beta \le k) \qquad (3.12)
$$

In according to this information, $p(t)$, for $\beta \to k$ the Caputo derivative become a conventional k-th derivative of the function $p(t)$, (see [71]). In addition to this,

$$
\lim_{\beta\to k} {}_a^C\mathfrak{D}_t^{\beta} p(t) = \lim_{\beta\to k} \left(\frac{p^{(k)}(\beta)(t-a)^{k-\beta}}{\Gamma(k-\beta+1)}\right) + \frac{1}{\Gamma(k-\beta+1)} \int_a^t (t-\tau)^{k-\beta} p^{(k+1)}(\tau)d\tau
$$

$$= p^{(k)}(\beta) + \int_{a}^{t} p^{(k+1)}(\tau)d\tau = p^{(k)}(t), \quad k = 1,2, \dots$$
$$= p^{(k)}(\beta) + {}_{a}I_{t}p^{(k+1)}(\tau)d\tau = p^{(k)}(t) \qquad (3.13)$$

This suggests that, similar to the Grunwald-Letnikov and Riemann-Liouville methods, the Caputo approach also provides a way to interpolate between integer-order derivatives [71]. A key benefit of the Caputo method is that the initial conditions for fractional differential equations using Caputo derivatives align with those for integer-order differential equations, involving the limit values of integer-order derivatives of the functions at the lower bound $t = a$, [71]. To highlight the differences in initial conditions between fractional differential equations with Riemann-Liouville and Caputo derivatives, we can examine the corresponding Laplace transform formulas for the case where $a = 0$ [71]. The Laplace transform for the Riemann-Liouville fractional derivative is given by [71].

$$\int_{0}^{\infty} e^{-st}\left\{{}_{0}\mathfrak{D}_{t}^{\beta}p(t)\right\}dt = s^{\beta}F(s) - \sum_{r=0}^{k-1} s^{r}\, {}_{0}\mathfrak{D}_{t}^{\beta-r-1}p(t)|_{t=0}, \qquad (k-1 \le \beta < k) \qquad (3.14)$$

The Laplace transform of the Caputo fractional derivative is done by, (see [71]-[72]);

$$\int_{0}^{\infty} e^{-st}\left\{{}_{0}^{C}\mathfrak{D}_{t}^{\beta}p(t)\right\}dt = s^{\beta}F(s) - \sum_{r=0}^{k-1} s^{\beta-r-1}\, p^{(r)}(0), \qquad (k-1 < \beta \le k) \qquad (3.15)$$

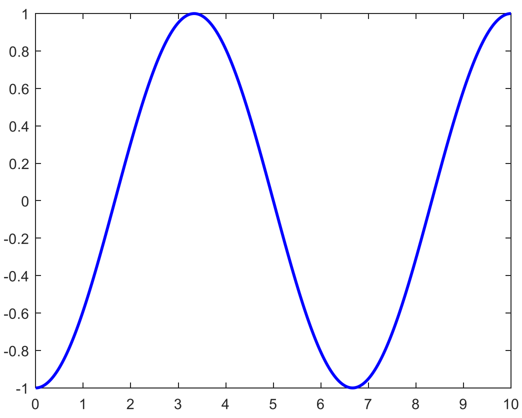
We applied the Laplace transform technique to both the Riemann-Liouville and Caputo fractional operators.

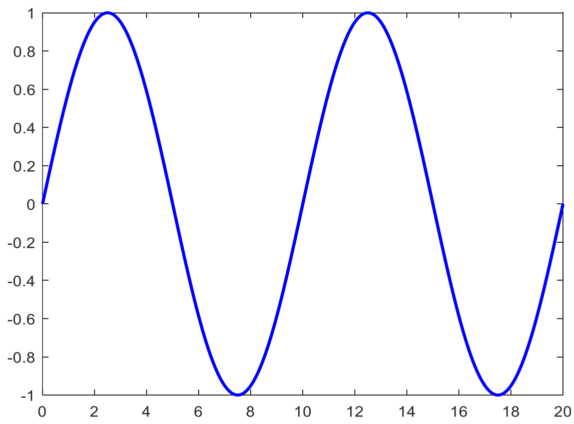
IV. Types of Waves and Wave Equations

4.1 Types of waves and wave equations

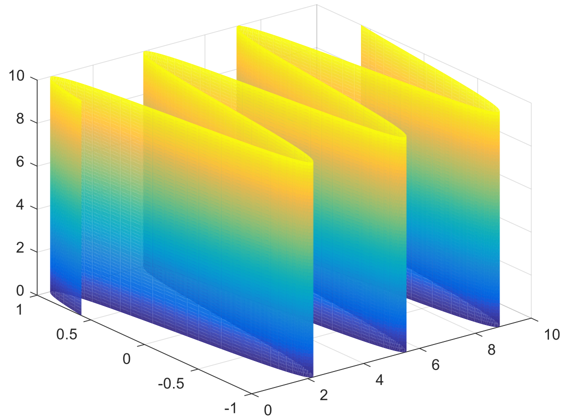
4.1.1 Transverse Wave

In physics, transverse waves oscillate perpendicular to their direction of travel [81]. Conversely, longitudinal waves oscillate parallel to their direction of movement, particularly when traveling along a surface [81]. Both wave types can propagate on surfaces; for example, seismic waves such as S-waves, P-waves, and Rayleigh waves all occur during an earthquake [81]. Electromagnetic waves, which are inherently transverse, do not require a medium for propagation [76]-[77]. The term "transverse" refers to waves where the oscillation direction is at right angles to the direction of particle movement in a medium or, for electromagnetic waves, perpendicular to the direction of wave propagation [78]-[79].





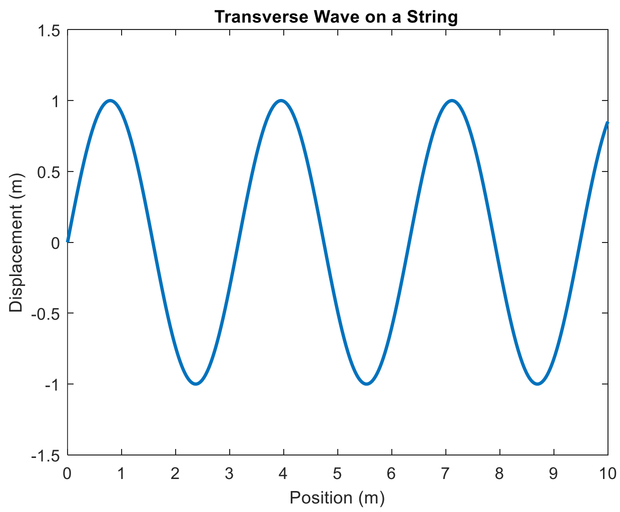
Here,
A: Amplitude of the wave.
k: Wave number, related to the wavelength (λ) by $k = \frac{2\pi}{\lambda}$.
ω: Angular frequency, related to the period (T) by $\omega = \frac{2\pi}{T}$.
t: Time variable, which changes to animate the wave.
x: Position along the wave.

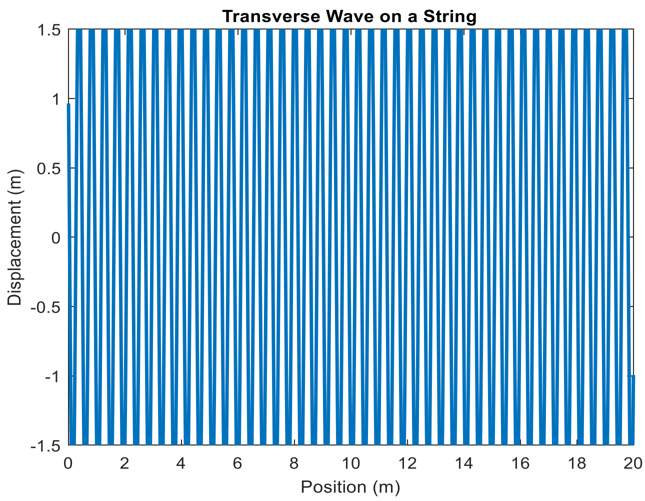


- **A**: The wave's amplitude.
- **k**: The wave number, which is connected to the wavelength.
- **ω**: The angular frequency.
- **t**: The time variable, which is updated within the loop to animate the wave.
- **X and Z**: 2D grids representing the coordinates along the x and z axes.

4.1.2 Mechanical Waves

Mechanical waves are disturbances that travel through a material medium, transmitting energy through it. Examples of mechanical waves include transverse waves, longitudinal waves, sound waves, and surface waves [80].



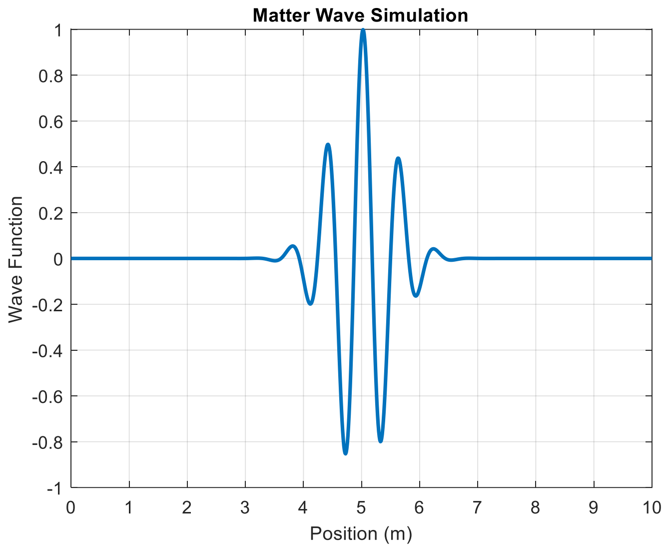


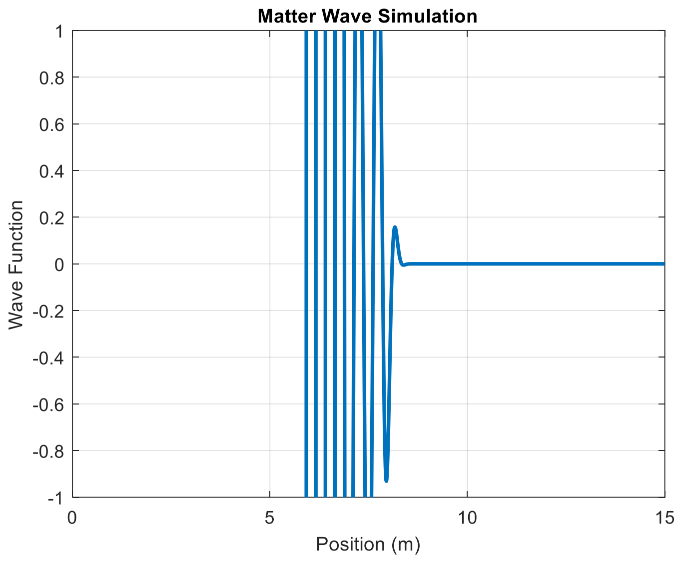
Wave Parameters:
L: The string's length.

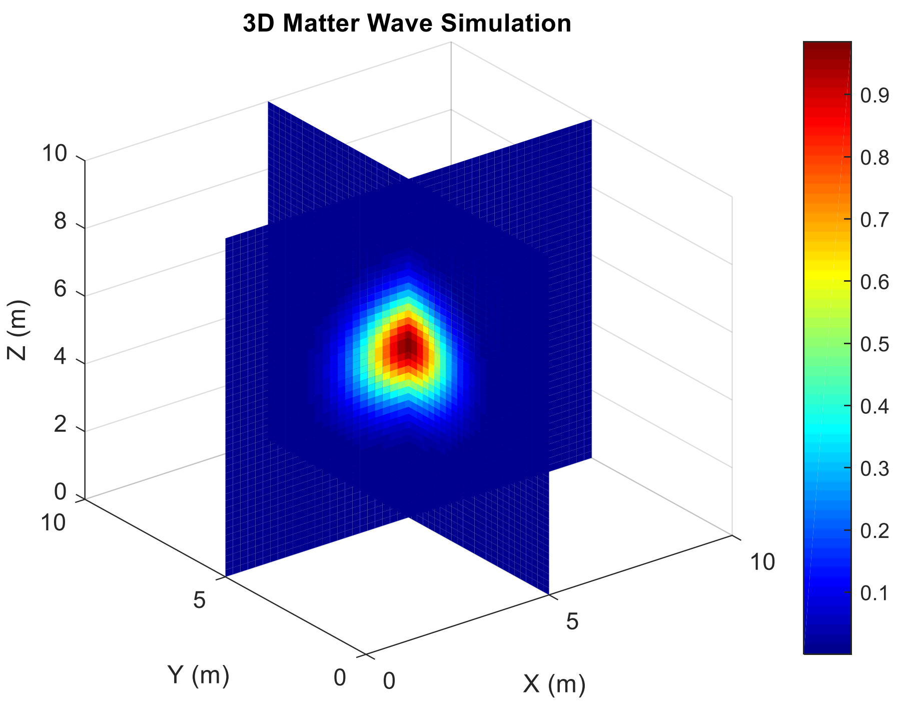
T: The tension within the string.
μ: The linear mass density, which is the mass per unit length of the string.
A: The wave's amplitude.
f: Frequency of the wave.
c: The speed of the wave, determined using the formula $c = \sqrt{T/\mu}$.
λ, k, ω: The wavelength, wave number, and angular frequency, respectively.

4.1.3 Matter Waves

In quantum physics, the concept of matter waves reveals that matter exhibits wave-like properties at scales where they can be observed [82]. For instance, electron beams, similar to light and water waves, can experience diffraction [82].





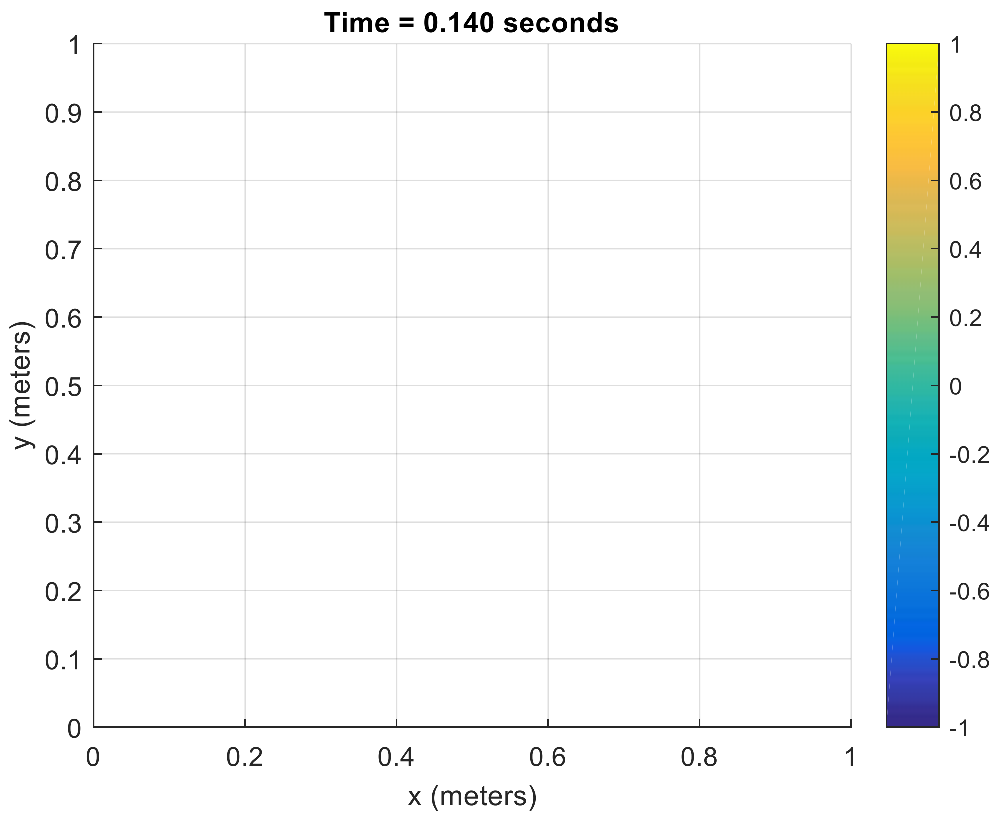


4.1.4 Sound Waves

In physics, sound describes vibrations that travel as acoustic waves through various media such as gases, liquids, or solids [83]. Sound can be understood as:

(a) Oscillations in pressure, stress, particle displacement, or particle velocity that propagate through a medium, influenced by internal forces such as elastic or viscous forces, or a combination of these [83].

(b)The sensory experience produced by these oscillations. Sound can be viewed as wave motion traveling through air or other elastic substances, serving as a stimulus, or as the activation of the auditory system that leads to the perception of hearing [83].



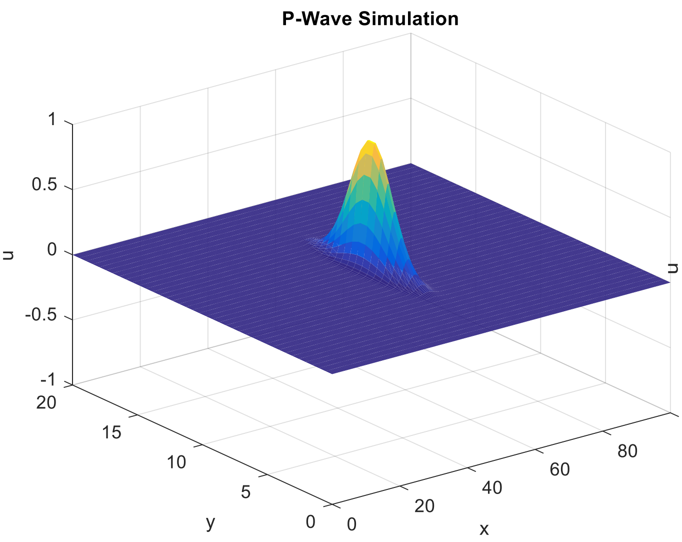
Explanation:

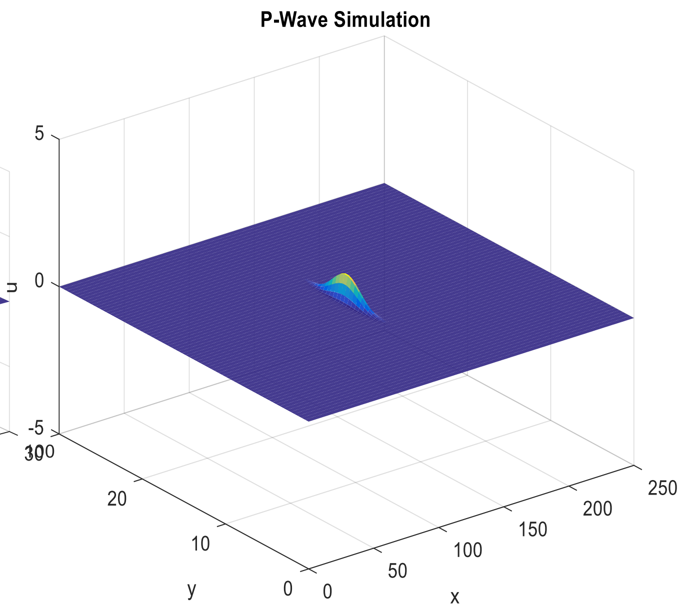
- **Parameters**: Establish the physical dimensions and settings for the simulation.
- **Initialization**: Create the 2D spatial grid and set up the initial pressure fields.
- **Source Term**: Specify a point source situated at the center of the grid.
- **Time-stepping Loop**: Update the pressure field with the 2D finite difference method and incorporate the source term.
- **Plotting**: Display the wave propagation in 2D through a surface plot.

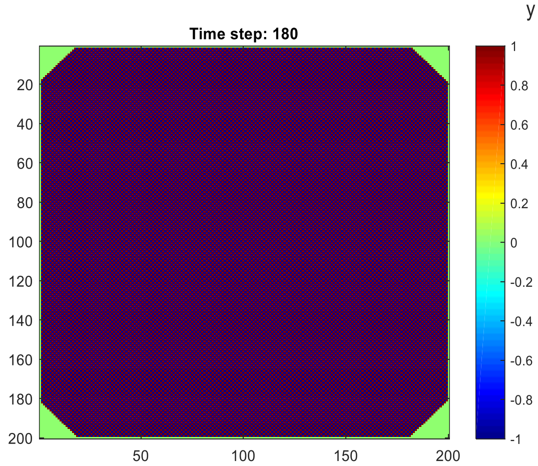
4.1.5 Surface Waves

Surface waves are seismic waves that originate at the Earth's surface and travel more slowly than other types of seismic waves. The main types of surface waves studied are Rayleigh and Love waves. Rayleigh waves travel along the surface of solids as acoustic waves and can be generated by localized impacts or piezoelectric methods. They are often used in non-destructive testing to detect defects and are also generated during earthquakes. When Rayleigh waves are confined to specific layers, they are referred to as Lamb waves, Rayleigh-Lamb waves, or generalized Rayleigh waves.

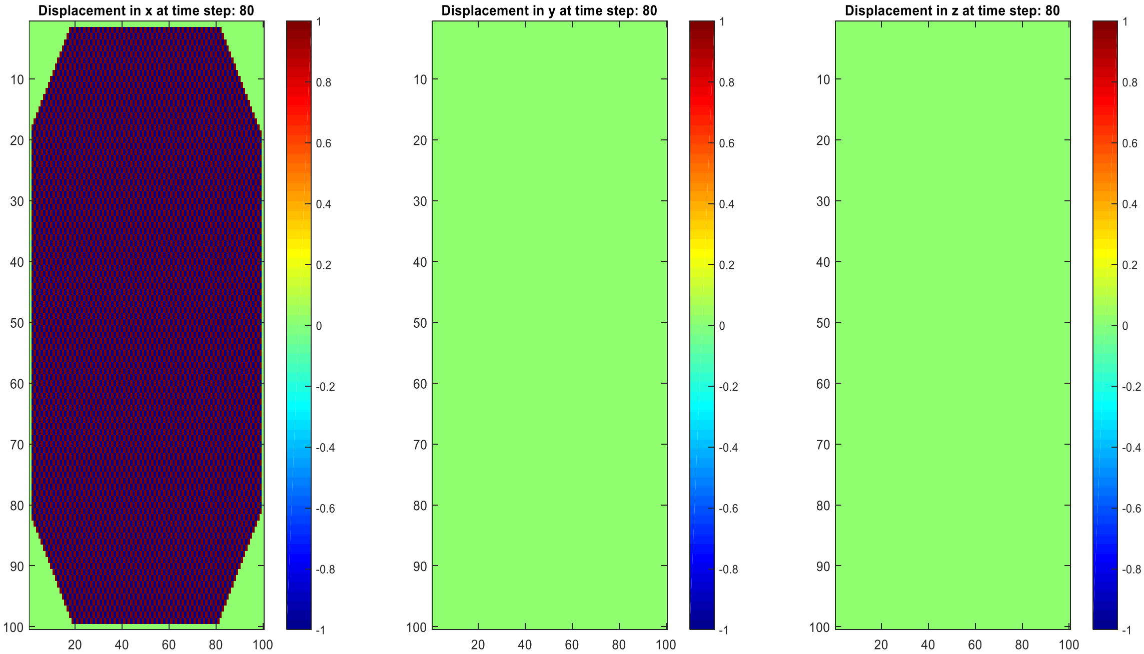
Love waves, named after Augustus Edward Hough Love, are horizontally polarized surface waves resulting from the interaction of shear waves (S-waves) guided by an elastic layer that is bonded to a half-space of elastic material, with a vacuum on the opposite side. In seismology, Love waves, also known as Q-waves (from the German word “Quer,” meaning lateral), cause horizontal displacement of the Earth's surface during earthquakes. Love waves, first mathematically predicted by Augustus Love in 1911, differ from other seismic waves such as P-waves and S-waves (body waves) and Rayleigh waves (another type of surface wave). While Love waves travel slower than P- and S-waves, they move faster than Rayleigh waves and occur when a low-velocity layer is situated above a high-velocity layer or sub-layers.







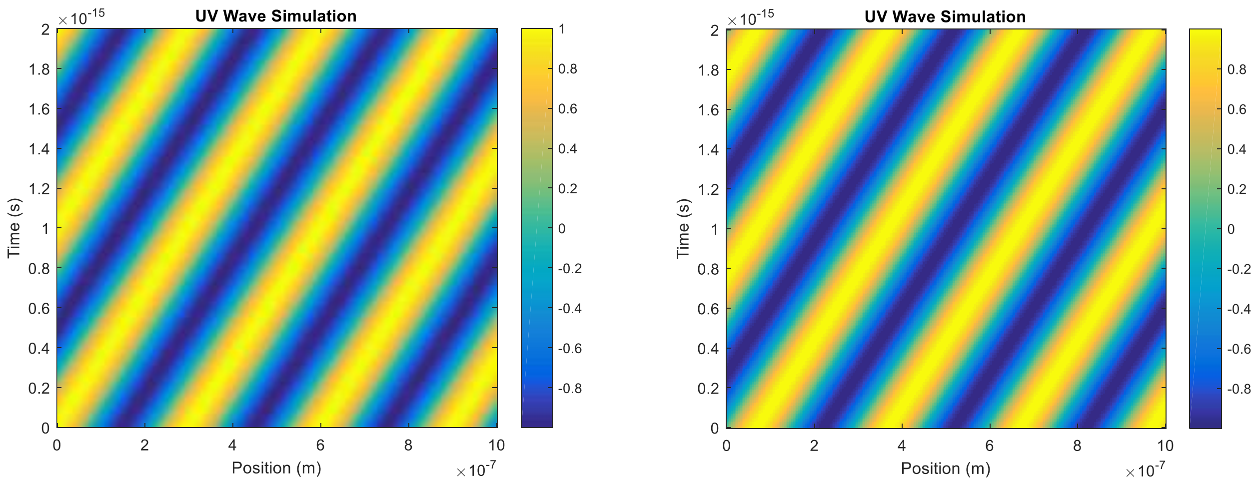
s-waves simulation in 2-dimension.



s-waves simulation in 3-dimension.

4.1.6 Ultraviolet

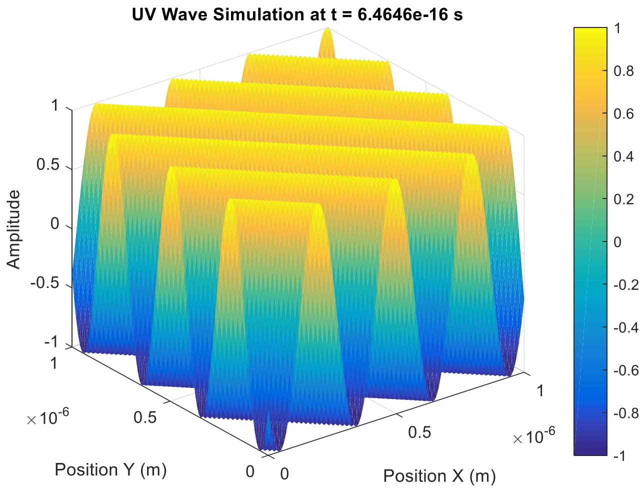
Ultraviolet (UV) light is electromagnetic radiation with wavelengths ranging from 10 to 400 nanometers, falling between visible light and X-rays [85]. UV radiation constitutes about 10% of the Sun's total electromagnetic output and can also be generated by electric arcs, Cherenkov radiation, and various types of lamps, including mercury-vapor, tanning, and black lights [85].

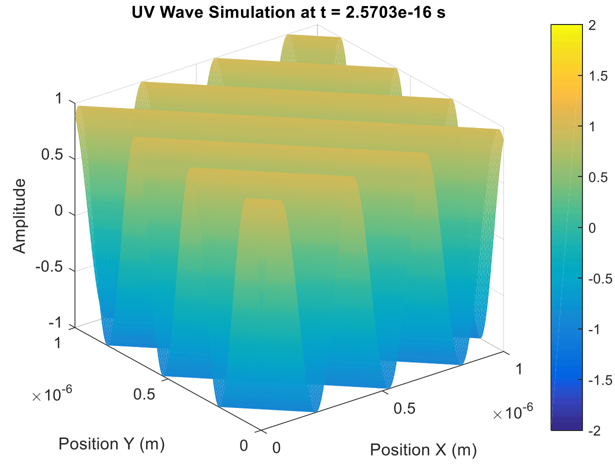


Wavelength and Frequency: The wavelength is set to 300 nm, which is within the ultraviolet range. The frequency is chosen accordingly.

Wave Number (k) and **Angular Frequency (ω)**: These are calculated using the wave parameters.

Wave Equation: The electric field $E(x, t)$ is computed as a cosine function representing the wave.





4.1.7 Progressive waves

Progressive waves, or traveling waves, are disturbances that carry energy through a medium without permanently altering the medium itself. As these waves move, the particles in the medium vibrate around their equilibrium positions but do not travel along with the wave [85]-[86].

- **Energy Transfer:** Progressive waves transmit energy from one point to another in the direction of wave propagation [85]-[86].
- **No Permanent Displacement:** After the wave passes, the particles return to their original positions, leaving the medium essentially unchanged [85]-[86].
- **Waveform Propagation:** The shape of the wave moves through the medium, with each point on the wave oscillating, though the phase of oscillation varies along the wave's direction [85]-[86].
- **Types:** Progressive waves can be either transverse, where vibrations are perpendicular to the direction of travel, or longitudinal, where vibrations are parallel to the direction of the wave [85]-[86].
- **Wave Equation:** The general equation for a progressive wave is [85]-[86].

$$y(x,t) = A\sin(kx - \omega t + \varphi)$$

where , A is the amplitude,

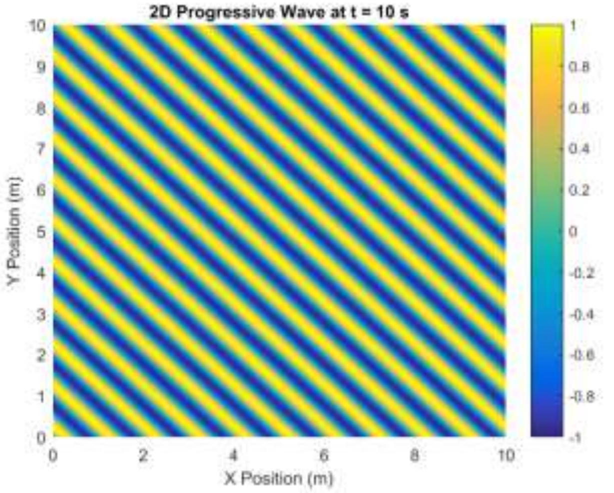
k is the wave number,

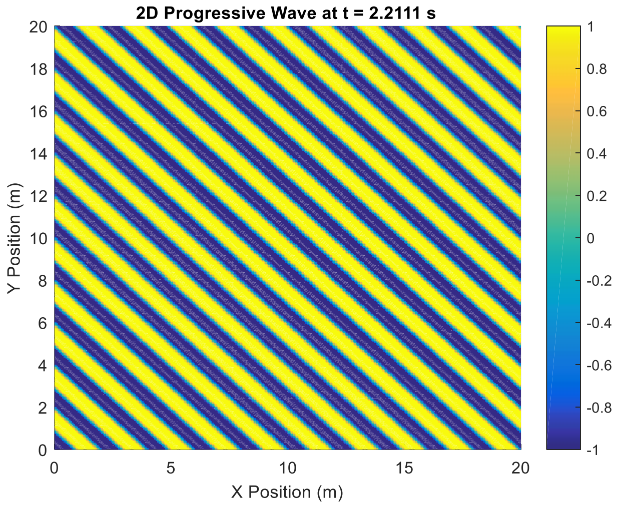
ω is the angular frequency,

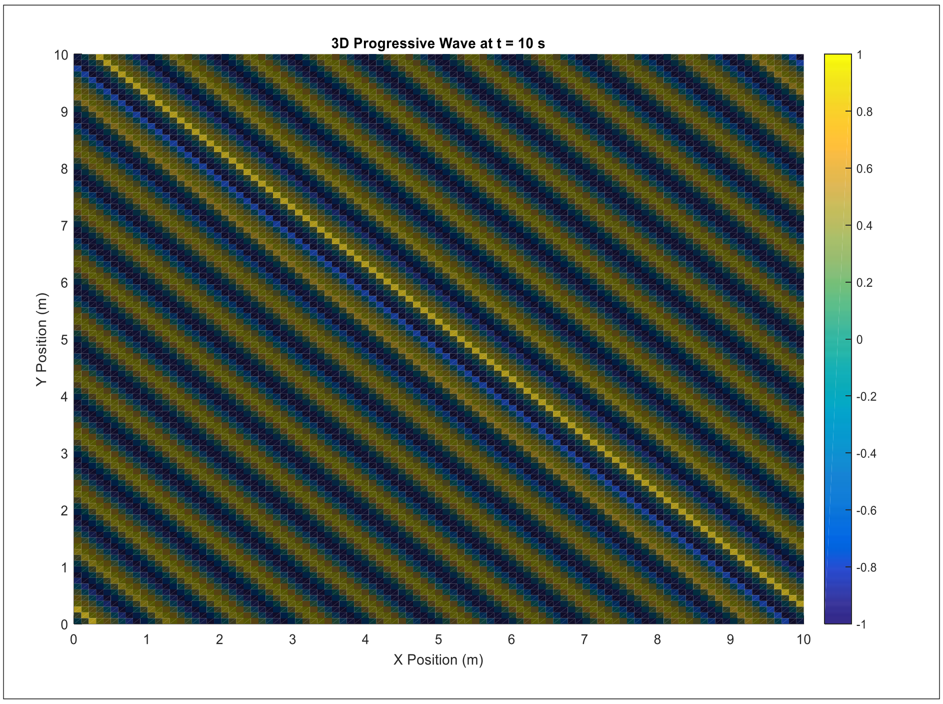
φ is the phase constant,

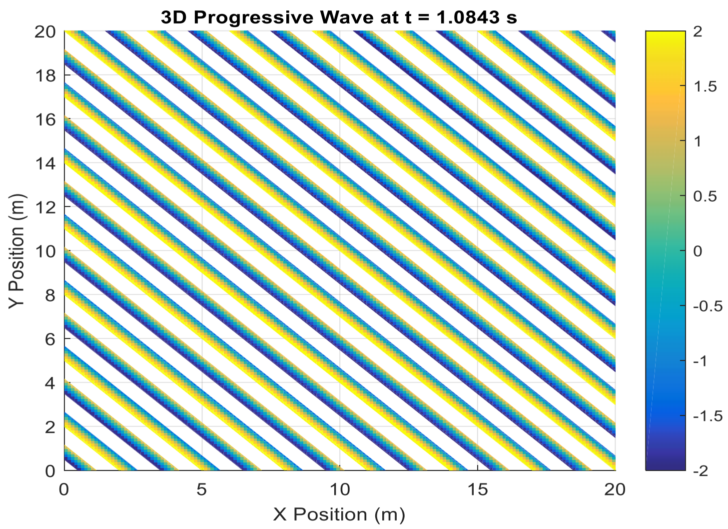
x is the position, and

t is the time.


2D Progressive Wave at t = 10 s
Y Position (m)
X Position (m)

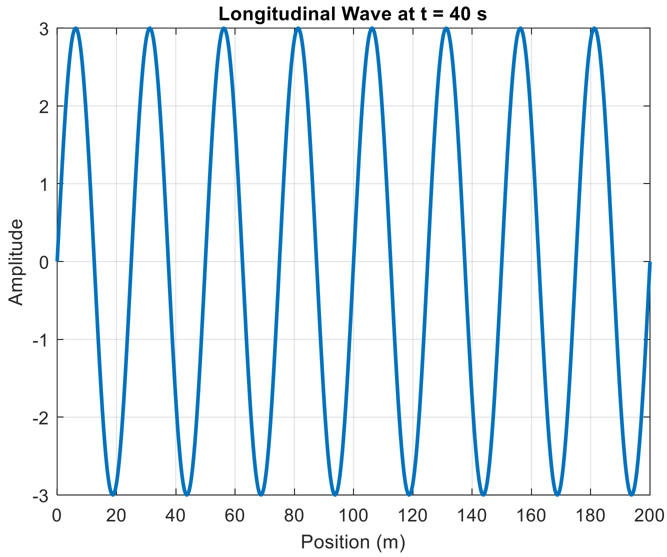

2D Progressive Wave at t = 2.2111 s
Y Position (m)
X Position (m)


3D Progressive Wave at t = 10 s
Y Position (m)
X Position (m)


3D Progressive Wave at t = 1.0843 s
Y Position (m)
X Position (m)

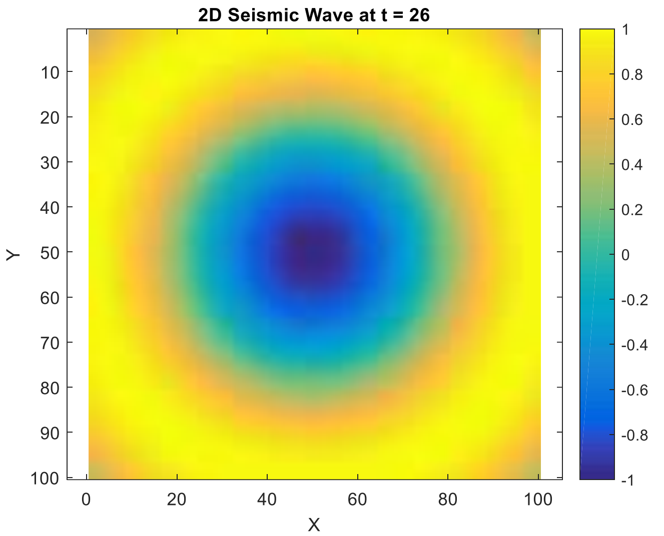
4.1.8 Longitudinal waves

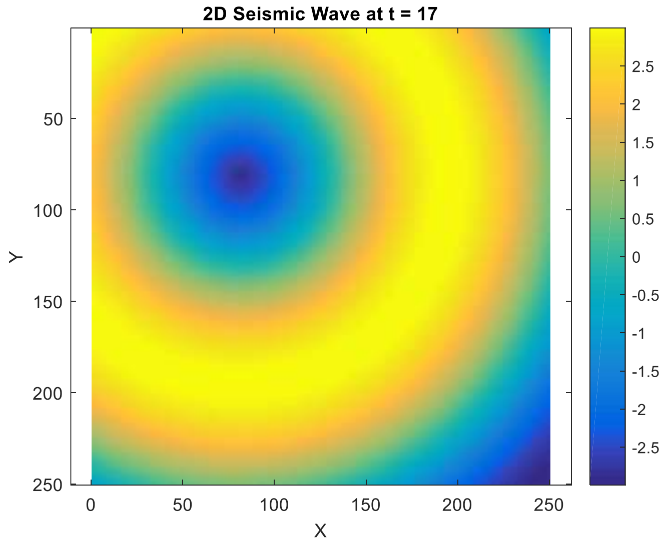
Longitudinal waves, also referred to as compressional or pressure waves, involve vibrations of the medium's particles in the same direction as the wave's travel [87]. This results in movement parallel to the wave's direction [87]. These waves generate alternating regions of compression and rarefaction, hence their name, and are also known for their impact on pressure changes [87]. An example is a wave traveling through a stretched Slinky toy, where the distance between the coils changes [87]. Common instances of longitudinal waves include sound waves, which involve pressure fluctuations and particle movement in elastic materials, and seismic P-waves, which occur during earthquakes [87].



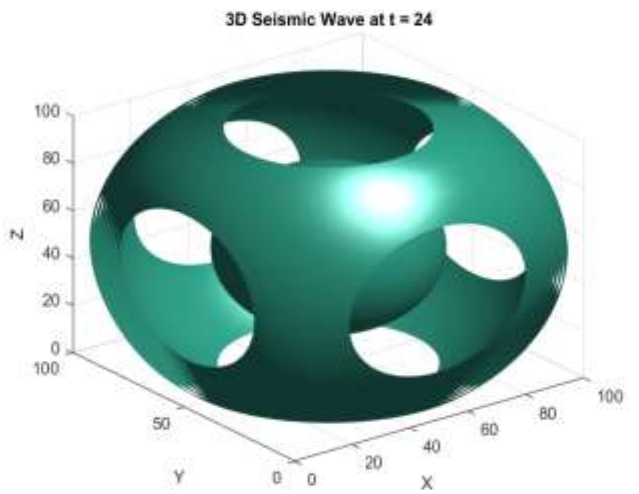
4.1.9 Seismic waves

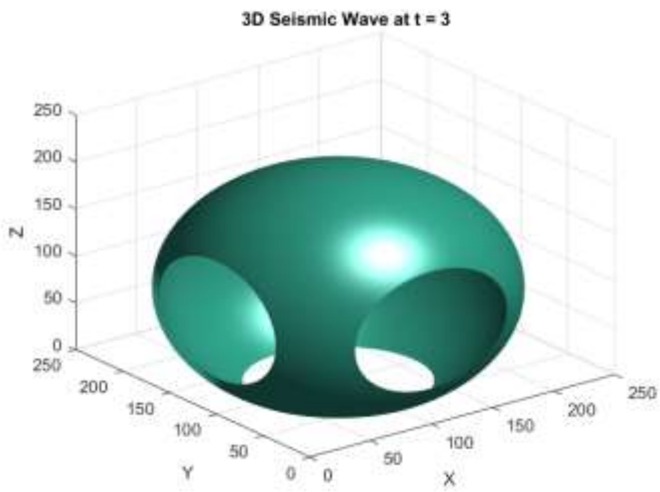
Seismic waves are mechanical waves that transmit acoustic energy through Earth or other celestial bodies [88]. These waves can be initiated by natural phenomena like earthquakes, volcanic activity, magma movements, and large-scale landslides, as well as by significant human activities such as explosions [88]. Seismologists employ instruments like seismometers, hydrophones (for underwater measurements), and accelerometers to analyze these waves [88]. Seismic waves differ from seismic noise, which consists of continuous low-amplitude vibrations from various sources [88]. The speed of seismic waves is influenced by the medium's density, elasticity, and the specific type of wave. Generally, wave velocities increase with depth in the Earth's crust and mantle, but they drop sharply when entering the outer core [89]. Earthquakes generate various types of waves, each traveling at different speeds [88]. By recording these waves, seismic observatories can help scientists pinpoint the earthquake's epicenter [88]. In geophysics, methods such as seismic wave refraction and reflection are vital for probing the Earth's interior, and researchers also generate and measure vibrations to explore shallow subsurface structures [88].





- **Grid Size**: Defines the 3D space where the wave will propagate.
- **Time Steps**: Sets the number of time steps for the simulation.
- **Wave Parameters**: Defines the wave's amplitude, speed, and the position of the wave source.
- **3D Grid**: A 3D matrix (x, y, z) is created to represent the space through which the wave propagates.
- **Wave Field Initialization**: The wave field is initialized to zero at the start.
- **Wave Propagation**: The loop calculates the distance from the source to each point in the grid and updates the wave field using a simple sinusoidal function.
- **Visualization**: The isosurface function is used to create a 3D visualization of the wave at each time step, with the drawnow command updating the plot in real time.

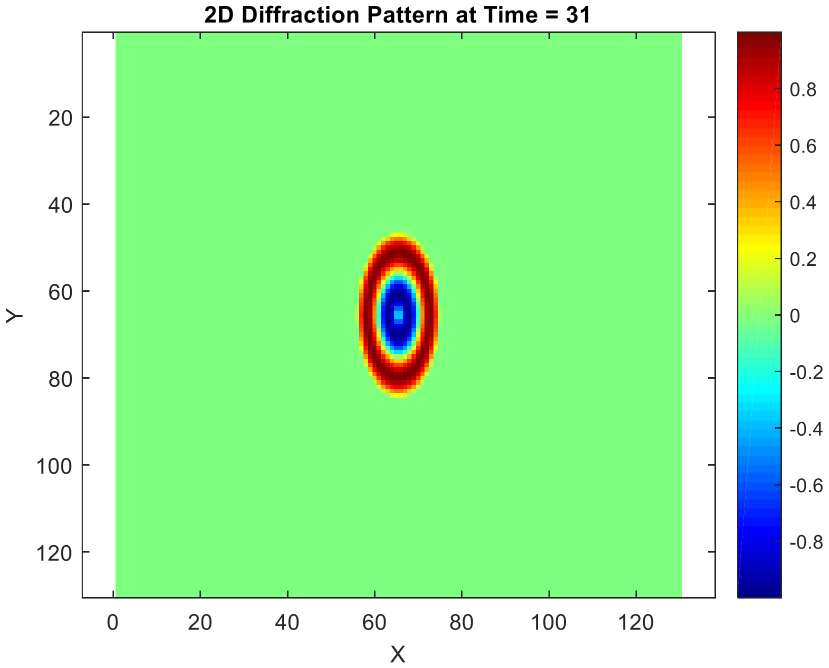




- **Grid Size**: Defines the 2D space where the wave will propagate.
- **Time Steps**: Sets the number of time steps for the simulation.
- **Wave Parameters**: Defines the wave's amplitude, speed, and the position of the wave source.
- **2D Grid**: A 2D matrix (x, y) is created to represent the space through which the wave propagates.
- **Wave Field Initialization**: The wave field is initialized to zero at the start.
- **Wave Propagation**: The loop calculates the distance from the source to each point in the grid and updates the wave field using a sinusoidal function.
- **Visualization**: The imagesc function displays the wave field as a color image, with colorbar showing the amplitude scale. The pause command creates an animation effect by updating the plot at each time step.

4.1.10 Diffraction

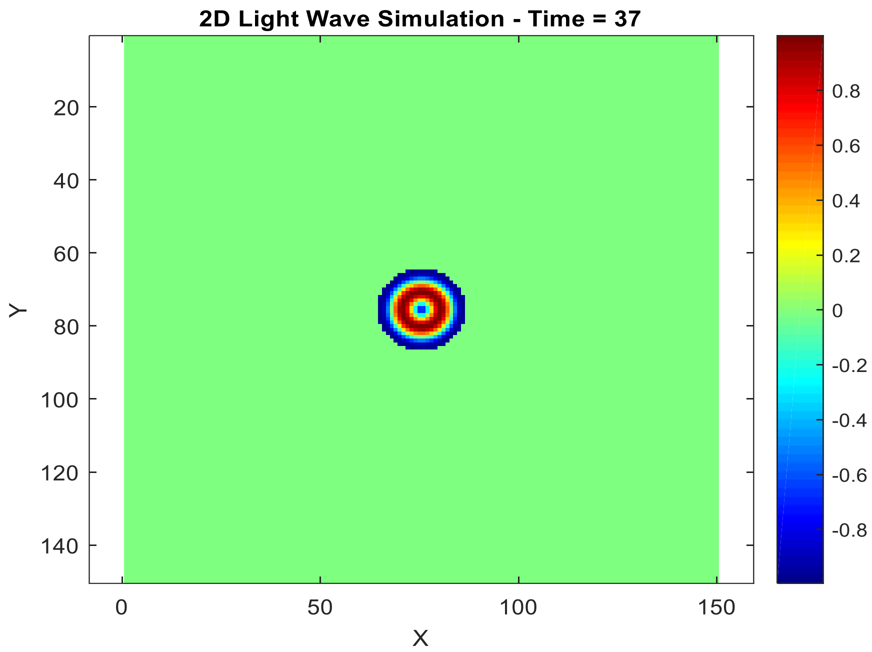
Diffraction refers to the bending or interference of waves as they encounter obstacles or pass through openings, allowing them to reach regions that would otherwise be in the shadow of the obstacle or aperture [90]. This occurs because the obstacle or opening acts as a new source of the wave [90]. The phenomenon of diffraction was first documented by Italian scientist Francesco Maria Grimaldi, who accurately described it in 1660, [90].



- **Grid Size**: Defines the 2D space where the wave will propagate.
- **Time Steps**: Sets the number of time steps for the simulation.
- **Wave Parameters**: Defines the wave's amplitude, speed, and the position of the wave source.
- **2D Grid**: A 2D matrix (x, y) is created to represent the space through which the wave propagates.
- **Wave Field Initialization**: The wave field is initialized to zero at the start.
- **Wave Propagation**: The loop calculates the distance from the source to each point in the grid and updates the wave field using a sinusoidal function.
- **Visualization**: The imagesc function displays the wave field as a color image, with colorbar showing the amplitude scale. The pause command creates an animation effect by updating the plot at each time step.

4.1.11 Light Waves

Visible light is a form of electromagnetic radiation that is detectable by the human eye [91]. It spans wavelengths ranging from 400 to 700 nanometers (nm) and frequencies between 750 and 420 terahertz [92]. Positioned between infrared radiation, which has longer wavelengths and lower frequencies, and ultraviolet radiation, which has shorter wavelengths and higher frequencies, visible light is part of the larger spectrum known as optical radiation [93]-[94]. In physics, the term "light" may also refer to electromagnetic radiation across all wavelengths, not just those within the visible range [95]-[96]. This includes gamma rays, X-rays, microwaves, and radio waves. The primary characteristics of light include its intensity, direction of propagation, frequency or wavelength spectrum, and polarization. In a vacuum, light travels at approximately 299,792,458 meters per second, a fundamental constant in natüre [97]. Light exhibits both wave-like and particle-like properties, with photons serving as the massless particles responsible for transmitting electromagnetic radiation. The study of light, known as optics, is a vital area in modern physics. The Sun is the main natural light source on Earth, with fire historically providing another significant source, from ancient fires to modern kerosene lamps. Today, electric lighting has largely replaced firelight as the primary means of illumination [91].



Grid Size and Parameters: Defines the grid size and parameters for the aperture and wavelength.

2D Grid Creation: Uses meshgrid to create a 2D spatial grid for X and Y coordinates.

Aperture Definition: Creates a circular aperture in the grid.

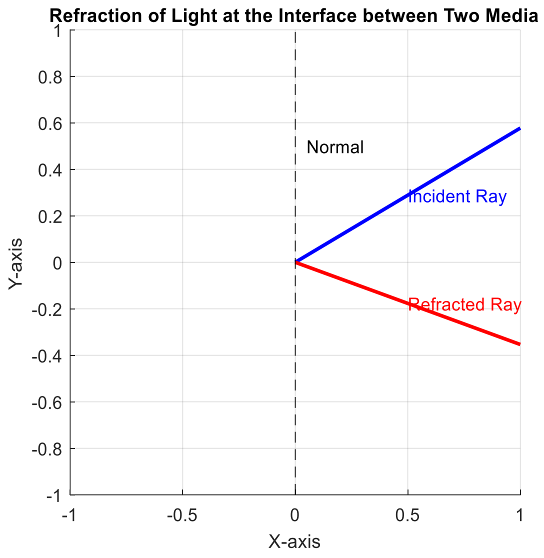
Wave Field Calculation: Simulates wave propagation and diffraction using a sine function.

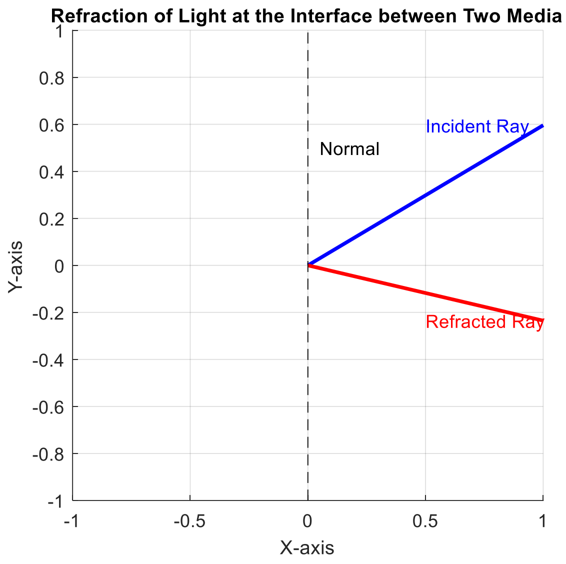
Aperture Effect: Applies the aperture effect by zeroing out the wave field outside the aperture.

Visualization: Uses imagesc to display the wave field as an image.

4.1.12 Refraction

In physics, refraction refers to the change in direction of a wave as it transitions from one medium to another [98]. This change in direction occurs due to a variation in the wave's speed, which is influenced by the properties of the new medium [99]. While light refraction is the most commonly observed example, other types of waves, such as sound waves and water waves, also undergo refraction [98]. The degree to which a wave is refracted depends on both the change in its speed and the angle at which it originally approached the boundary between the two media [98].



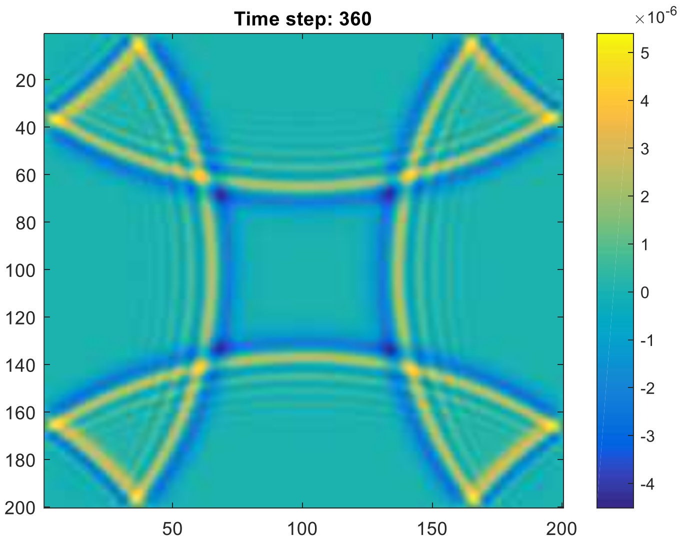


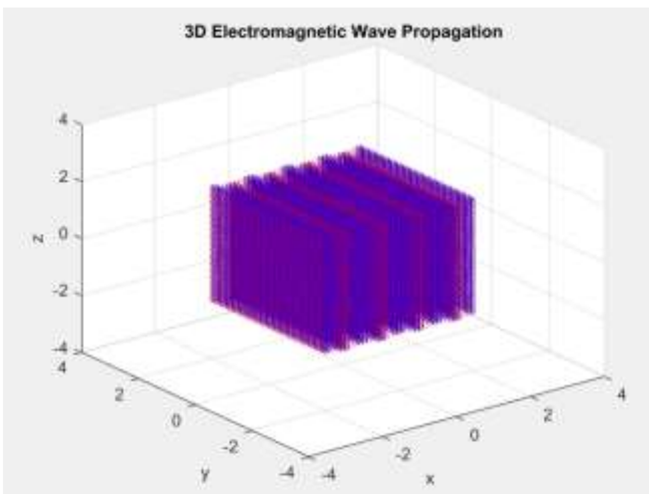
4.1.13 Electromagnetic waves

Electromagnetic (EM) waves consist of oscillating electric and magnetic fields that carry energy and momentum through space [100]. These waves are solutions to Maxwell's equations, the core principles of electrodynamics [100]. Unlike mechanical waves, EM waves do not require a medium and can propagate through a vacuum [100]. One example of an EM wave is the sinusoidal plane wave. Although not all EM waves are sinusoidal plane waves, any electromagnetic wave can be represented as a linear combination of sinusoidal plane waves moving in different directions [100]. A plane EM wave moving along the x-axis can be expressed in the following form [100].

$$E(x,t) = E_{max}\cos(kx - \omega t + \varphi),\quad B(x,t) = B_{max}\cos(kx - \omega t + \varphi).$$

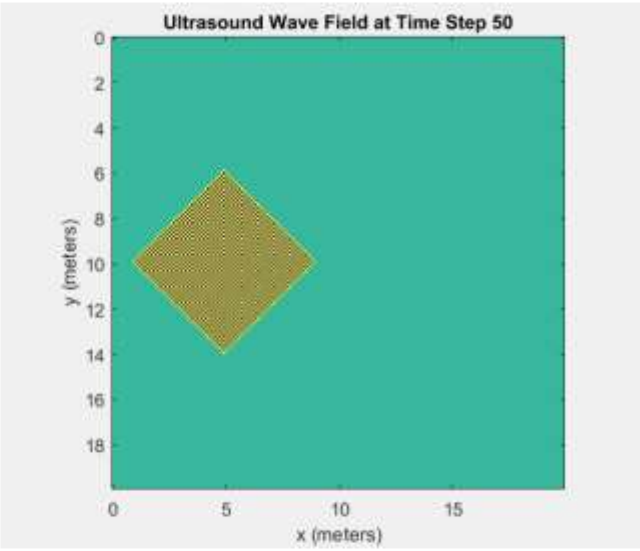
E is the electric field vector, and B is the magnetic field vector of the EM wave [100]. For electromagnetic waves E and B are always perpendicular to each other and perpendicular to the direction of propagation [100]. The direction of propagation is the direction of E x B [100]. "If an electromagnetic wave propagates in the x-direction with the electric field E oriented along the j-axis, then the magnetic field B will be along the k-axis, and the cross product $j \times k$ results in the unit vector i, [100]. Electromagnetic waves are characterized as transverse waves [100].

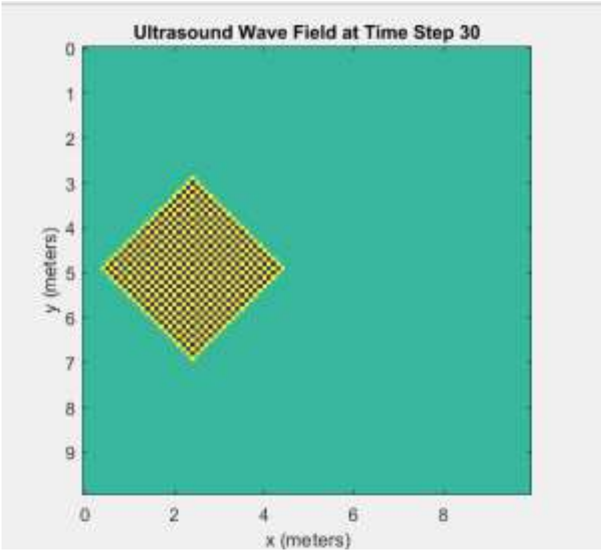




4.1.14 Ultrasound Waves

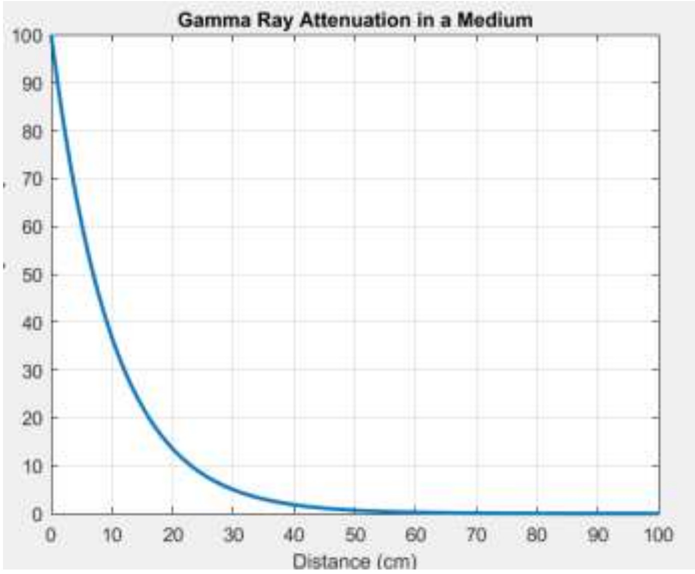
Ultrasound refers to sound waves with frequencies exceeding 20 kilohertz, which is the approximate upper limit of human hearing for young, healthy individuals [101]. The principles governing acoustic waves apply to all frequency ranges, including ultrasound [102]. Devices that use ultrasound operate within a frequency range from 20 kHz to several gigahertz [102]. Ultrasound technology has various applications [102]. It is utilized for object detection and distance measurement [102]. In the medical field, ultrasound imaging, also known as sonography, is commonly employed. In nondestructive testing, ultrasound helps identify hidden defects in products and structures [102]. Additionally, ultrasound is used in industrial processes for tasks like cleaning, mixing, and speeding up chemical reactions. Certain animals, including bats and porpoises, use ultrasound to navigate and locate prey and obstacles [103].

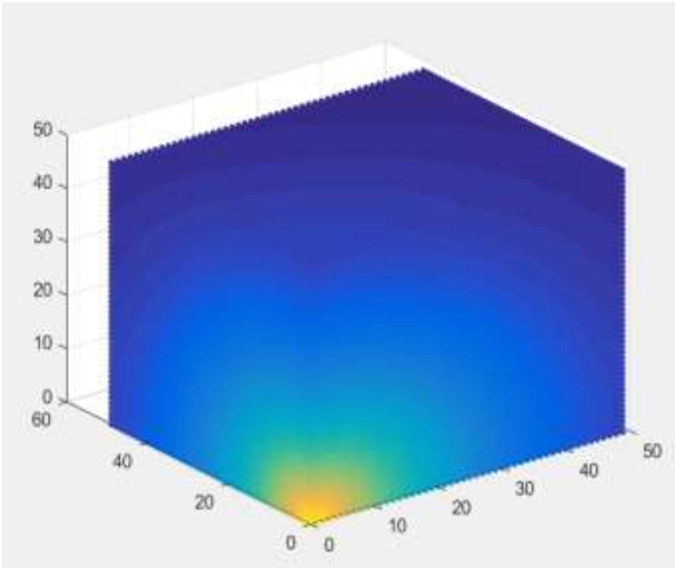




4.1.15 Gamma ray

Gamma rays, denoted by the symbol γ, represent a highly penetrating type of electromagnetic radiation produced during the radioactive decay of atomic nuclei [104]. These rays have the shortest wavelengths in the electromagnetic spectrum, typically shorter than X-rays [104]. Gamma rays possess frequencies exceeding 30 exahertz (3.10^{19} Hz) and wavelengths less than 10 picometers (10^{-11} m), making their photons the most energetic of all electromagnetic radiation [104]. The discovery of gamma radiation is attributed to Paul Villard, a French chemist and physicist, who identified it in 1900 while examining radiation from radium [104]. Subsequently, in 1903, Ernest Rutherford coined the term "gamma rays" for this radiation due to its significant penetrating ability, distinguishing it from the alpha and beta rays discovered earlier by Henri Becquerel, which were named in order of increasing penetration strength [104]. Gamma rays emitted from radioactive decay typically have energies ranging from a few kiloelectronvolts (keV) up to around 8 megaelectronvolts (MeV), which aligns with the energy levels of nuclei with relatively long lifetimes [104]. Gamma spectroscopy can analyze this energy spectrum to determine the identity of decaying radionuclides [104]. Extremely high-energy gamma rays, with energies between 100 and 1000 teraelectronvolts (TeV), have been detected from cosmic sources like the Cygnus X-3 microquasar [104]. On Earth, gamma rays primarily arise from radioactive decay and secondary radiation due to interactions between cosmic rays and the atmosphere [104]. Additionally, rare natural phenomena, such as terrestrial gamma-ray flashes, occur when electrons interact with nuclei, producing gamma rays [104]. Artificial sources of gamma rays include processes like nuclear fission in reactors and high-energy physics experiments, such as the decay of neutral pions and nuclear fusion [104]. Gamma rays are a form of ionizing radiation and can be harmful to living organisms [104]. They have the potential to induce DNA mutations, increase the risk of cancer and tumors, and cause severe effects like burns and radiation sickness at high exposures [104]. Their significant penetrating ability allows them to affect internal organs and bone marrow [104]. In contrast to alpha and beta radiation, gamma rays can traverse the body with relative ease, which presents a significant challenge for radiation protection [104]. Effective shielding often requires dense materials such as lead or concrete [104]. On Earth, the magnetosphere shields most forms of dangerous cosmic radiation, though it provides limited protection against gamma rays [104].



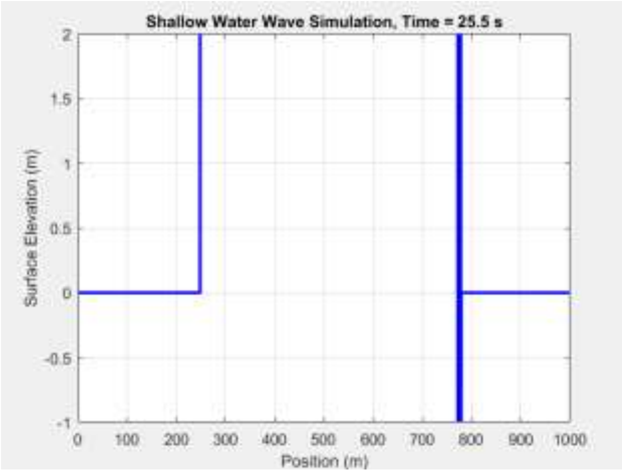


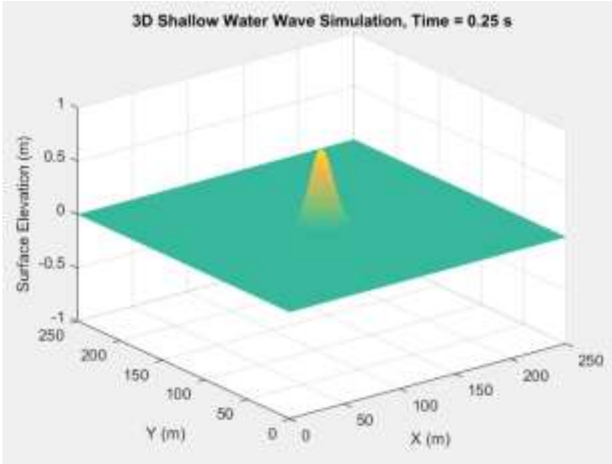
4.1.16 Microwave

Microwaves are a type of electromagnetic radiation characterized by wavelengths that are shorter than traditional radio waves but longer than infrared radiation [105]. Their wavelength spans approximately from one meter to one millimeter, which corresponds to frequency ranges between 300 MHz and 300 GHz, [105].

4.1.18 Shallow water wave

As waves move into shallow water, they start to interact with the ocean floor, which disrupts their free orbital motion [107]. The water particles no longer return to their initial positions as they would in deeper water. In shallower areas, the waves become taller and more abrupt, eventually forming the characteristic sharp-crested shape [106]. Once the wave breaks, it transforms into a wave of translation, leading to increased erosion of the seabed [106]. Cnoidal waves are precise periodic solutions to the Korteweg–de Vries equation, applicable in shallow water where the wavelength significantly exceeds the water depth [106].





4.1.19 Wave Equation

In physics, the wave equation plays a crucial role in providing a mathematical interpretation of wave phenomena. Although it is often studied within the reals of theoretical physics and classical mechanics, our focus here is on its mathematical formulation. A wave can be described as a function that represents the propagation of waves across surfaces, whether in two dimensions or higher. By considering the one-dimensional case, the wave equation can be derived from Maxwell's equations [108]. For scalar functions, then,

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \qquad (4.1)$$

for light wave and c is a light speed. In reference [108], the equations for water waves, air waves, and seismic waves describe the properties of most wave phenomena, extending beyond just light waves. In many practical scenarios, the velocity of a wave varies with its amplitude, meaning that $v = v(f)$. This relationship can make finding solutions more challenging. Suppose $v_1(x,t)$ and $v_2(x,t)$ are solutions; according to D'Alembert's equation, the wave equation can be expressed with v as the characteristic. The characteristic equation [109] is given by $x \pm ct$ = constant, where the $\pm$ sign represents the two solutions to the quadratic equation [109]. Therefore, a variable transformation can be applied, $\xi = x + ct$ for positive solution, $\gamma = x - ct$ for negative solution, [109], and $u_{\xi\gamma} = u_{\gamma\xi} = 0$. The general solution of partial differential equation $u(\xi,\gamma) = \psi(\xi) + \varphi(\gamma)$ where φ and ψ are C^1 functions, [109]. Let's look x, t coordinates,

$$u(x,t) = \psi(x+ct) + \varphi(x+ct)$$

u is C^2if ψ and φ are C^2. For,

$u(x,0) = \psi(x) + \varphi(x) = k(x)$ and $u_t(x,0) = c\psi'(x) - c\varphi(x) = c\{\psi'(x) - \varphi(x)\} = p(x)$. We can integrate the last equation to get (see [109]);

$$c\psi(x) - c\varphi(x) = \int_{-\infty}^{x} p(\xi)d\xi + c_1, \qquad c_1 \in \mathbb{R}.$$

and we can solve the system using known equations to obtain [109],

$$\varphi(x) = -\frac{1}{2c}\left(-ck(x) - \left(\int_{-\infty}^{x} p(\xi)d\xi + c_1\right)\right)$$

$$\psi(x) = -\frac{1}{2c}\left(-ck(x) + \left(\int_{-\infty}^{x} p(\xi)d\xi + c_1\right)\right)$$

Now, using [109];

$$u(x,t) = \psi(x+ct) + \varphi(x-ct)$$

accordingto the D'Alembert's formula becomes,

$$u(x,t) = \left\{\frac{k(x+ct) + k(x-ct)}{2}\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} p(\xi)d\xi$$

$$u(x,t) = \frac{1}{2}\{k(x+ct) + k(x-ct)\} + \frac{1}{2c}{}_{x-ct}I_{x+ct}p(\xi)d\xi \tag{4.2}$$

The numerical resolution of the wave equation is a significant topic in computational mathematics and physics, with diverse applications such as seismic wave analysis and electromagnetic wave simulations. The Finite Difference Method (FDM) is a widely used numerical technique for addressing the wave equation, known for its straightforwardness and efficiency. Here's a summary of how FDM is applied to the wave equation, along with essential concepts and references for additional study.

Finite Difference Method for the Wave Equation

Formulation

The wave equation in one dimension is given by [109]-[110]-[111]-[112]-[113]-[114]-[115]-[116]-[117]-[118]-[119]-[120]-[121]-[122],

$$v_{tt} = c^2 v_{xx}$$

To apply the finite difference method, discretize the spatial and temporal domains:

Spatial Domain: Divide the spatial domain $x \in [0, L]$ into N intervals of length Δx resulting in a grid of points $x_i = i\Delta x$, where $i = 0,1,2,\dots,N$.

Temporal Domain: Divide the time interval $t \in [0,T]$ into M intervals of length Δt, resulting in time steps $t^n = n\Delta t$, where $n = 0,1,\dots,M$.

Discretization: We need to apply finite differences to approximate the derivatives:

i) Second derivative in time:

$$\frac{\partial^2 v(x_i, t^n)}{\partial t^2} \approx \frac{v_i^{n+1} - 2v_i^n + v_i^{n-1}}{(\Delta t)^2}$$

ii) Second derivative in space:

$$\frac{\partial^2 v(x_i, t^n)}{\partial x^2} \approx \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2}$$

Approximatioons substitution is used into the wave equation:

$$\frac{v_i^{n+1} - 2v_i^n + v_i^{n-1}}{(\Delta t)^2} = c^2 \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2}$$

Rearranging to solve for u_i^{n+1}:

$$v_i^{n+1} = 2v_i^n - v_i^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 (v_{i+1}^n - 2v_i^n + v_i^{n-1})$$

1) Dirichlet Boundary Conditions

These conditions specify the value of the wave function at the boundaries:

$$u(0, t) = U(L, t) = 0$$

2) Neumann Boundary Conditions

These conditions fix the gradient of the wave function at the boundaries:

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$$

3) Robin Boundary Conditions

Robin conditions combine Dirichlet and Neumann conditions:

$$\alpha u(0, t) + \beta \frac{\partial u}{\partial x}(0, t) = 0$$

$$\alpha u(L, t) + \beta \frac{\partial u}{\partial x}(L, t) = 0$$

4) Periodic Boundary Conditions

These assume the wave function is periodic:

$$u(0, t) = u(L, t)$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t)$$

5) Mixed Boundary Conditions

$$u(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0.$$

Finite Element Method for the Wave Equation

Finite Element Method (FEM) is a versatile technique for numerically solving the wave equation, particularly useful for complex geometries and boundary conditions.

6) Overview of FEM for the Wave Equation

1. Formulation

Transform the wave equation into its weak form by multiplying with a test function and integrating:

$${}_0I_{L,x}\{u_{tt}v(x)\} + c^2\,{}_0I_{L,x}\{u_x v_x\} = 0$$

We offer a comprehensive analysis of the numerical outcomes obtained from applying the Finite Element Method (FEM) to wave equations. To illustrate this, we will explore a specific case. Our focus will be on solving a one-dimensional (1D) wave equation using FEM, followed by an interpretation of the results. This process includes discretization, solving the equation, and analyzing the results [123]-[124]-[125]-[126]-[127]-[128].

Problem definition

Consider the 1D wave equation (see [129]-[130]-[131]-[132]-[133]-[134]):

$$\frac{\partial^2 v(x,t)}{\partial t^2} = c^2 \frac{\partial^2 v(x,t)}{\partial x^2}, \quad 0 \le x \le L, \quad t \ge 0.$$

with the following boundary and initial conditions:

- Boundary conditions: $v(0,t) = v(L,t) = 0$, (Dirichtlet conditions)
- Initial conditions: $v(x,0) = f(x)$ and $\frac{\partial v(x,0)}{\partial t} = g(x)$ where $f(x)$ represents the initial displacement, and $g(x)$ represents the initial velocity.

Numerical Solution Using Finite Element Method

1. Discretization

. Spatial Discretization: The interval $[0,L]$ is divided into N elements, each with a length $h = \frac{L}{N}$. The nodes are positioned at $x_i = i.h$, where $i = 0,1,\ldots N$, ([129]-[130]-[131]-[132]-[133]-[134]).

. Temporal Discretization: Time is divided into steps of size Δt. For simplicity, the Newmark-beta method is employed for time integration, (see [129]-[130]-[131]-[132]-[133]-[134]).

2. Element Matrices

For each element, the local stiffness matrix K_e and mass matrix M_e are calculated. For linear shape functions in a 1D element with nodes i and $i + 1$, (see [129]-[130]-[131]-[132]-[133]-[134]):

$$K_e = \frac{c^2}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$M_e = \frac{h}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

3. Assembly

The global stiffness K and mass M matrices are formed by summing contributions from all elements. For N nodes, these matrices are of size $N \times N$, (see [129]-[130]-[131]-[132]-[133]-[134]).

4. Application of Boundary Conditions

Boundary conditions are enforced by modifying the global matrices accordingly. Since $u(0, t) = u(L, t) = 0$, the corresponding rows and columns in the matrices are adjusted, (see [129]-[130]-[131]-[132]-[133]-[134]).

5. Time Integration

Using the Newmark-beta method (with parameters β and γ), the wave equation is solved iteratively over time. The displacement at each time step is calculated as, (see [129]-[130]-[131]-[132]-[133]-[134]):

$$v_{n+1} = v_n + \Delta t \dot{v}_n + \frac{\Delta t^2}{2}\ddot{v}_n$$

where $v_n, \dot{v}_n, \ddot{v}_n$ are the displacement, velocity, and acceleration at time t_n.

Example and Numerical Results

Let's examine a specific example:

- **Domain:** $L = 1$ (unit length)
- **Wave speed:** $c = 1$ (unit wave speed)
- **Initial condition:** $u(x, 0) = sin(\pi x), \frac{\partial u(x,0)}{\partial t} = 0$
- **Discretization:** $N = 10$ elements, $\Delta t = 0.01$

After assembling the global matrices and applying the initial and boundary conditions, the FEM solution progresses over time. The results can be summarized as follows, (see [129]-[130]-[131]-[132]-[133]-[134]):

- **Initial State:** At $t = 0$, the displacement $u(x, 0)$ is a sine wave, with maximum amplitude at $x = 0.5$
- **Wave Propagation:** Over time, the wave propagates toward the boundaries, reflecting off $x = 0$ and $x = 1$ due to the boundary conditions.

Time Evolution:

- $t = 0.1$: The wave begins to propagate, and the peak displacement shifts slightly toward the boundaries.
- $t = 0.2$: The wave front approaches the boundaries, and reflections start.
- $t = 0.3$: The wave reflects, forming a pattern similar to the initial condition but with the opposite phase.

Final State: At later times, the wave continues oscillating, reflecting between the boundaries with gradually decreasing amplitude due to numerical damping, (see [129]-[130]-[131]-[132]-[133]-[134]).

Error Analysis

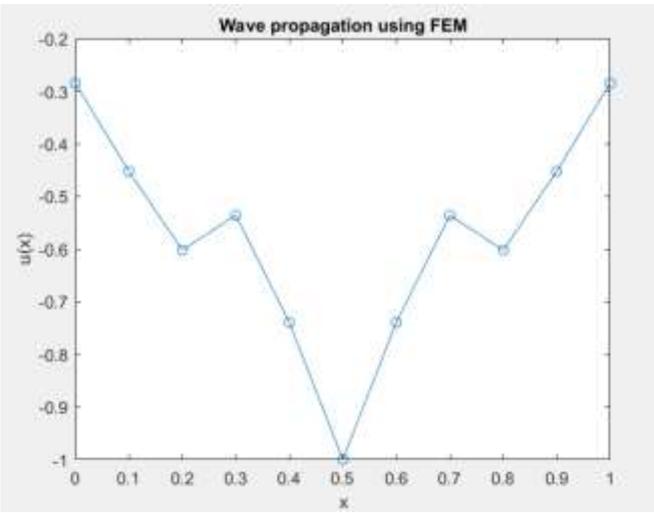
- **Mesh Convergence:** Increasing the number of elements N improves accuracy, bringing the solution closer to the exact wave equation solution.

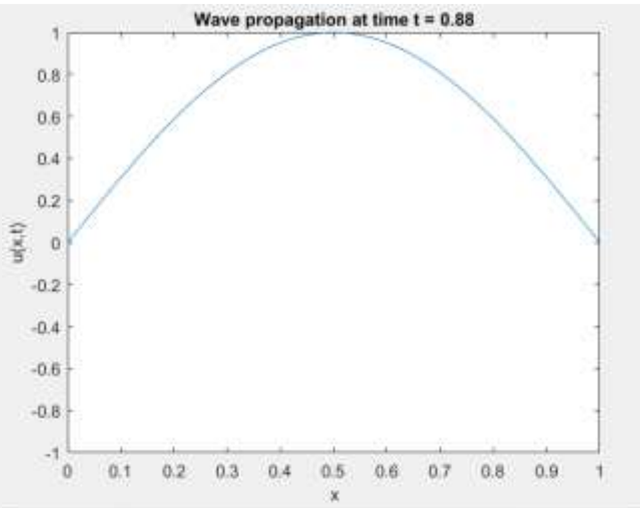
- **Time Step Sensitivity:** Accuracy also depends on the time step Δt. Smaller Δt increases accuracy but requires more computational effort.

Visualization

- **Displacement over Time:** Plotting $u(x, t)$ at different time steps shows the wave's propagation and reflection.
- **Error Comparison:** Comparing the FEM solution with an analytical solution (if available) helps assess the method's accuracy.

The finite element method is an effective and adaptable method for solving the wave equation, particularly for problems with complex boundary conditions or irregular geometries. The numerical results demonstrate the wave's propagation and interactions within the domain, providing valuable insights into the system's physical behavior, (see [129]-[130]-[131]-[132]-[133]-[134]).





4.2 Fractional wave equations

Now, we show fractional order wave equations according to the above information. But mathematical induction method is used to show step to step to introduce detail of fractional version of this equation which depends on x and t. First of all, our information was written $v(x \mp ct)$ and it solves the wave equation. According to the information, $v(x \mp ct)$ and

$$\frac{\partial u}{\partial x} = 1$$
$$\frac{\partial u}{\partial t} = \pm c$$

then, chain-rule is necessary to demonstrate, fractional order wave equation with mathematical induction method.

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial x}$$
$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial v}{\partial u}(\pm c)$$
$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial u^2}, \quad \frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial u^2}$$
$$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2}$$
$$\frac{\partial^3 v}{\partial t^3} = c^3\frac{\partial^3 v}{\partial x^3}$$

•

•

•

$$\frac{\partial^\alpha v}{\partial t^\alpha} = c^\alpha\frac{\partial^\alpha v}{\partial x^\alpha}, \quad 0 < \alpha < 1, \qquad (4.3)$$

then (4.3) is fractional order wave equation.

4.2.1 Overview of Fractional Wave Equations

Fractional wave equations extend classical wave models by incorporating fractional derivatives, offering a more nuanced approach to describing physical processes, especially those involving memory effects or anomalous diffusion. Traditional wave equations rely on integer-order derivatives, which may not fully capture the complexity of some systems. Fractional derivatives provide a way to model such systems with greater accuracy. A general fractional wave equation can be expressed as [135]-[136]-[137],

$$D_t^\alpha u(x,t) = c^2\nabla^2 u(x,t) + f(x,t),$$

where,

. D_t^α denotes a fractional derivative of order α\alphaα (typically using Caputo or Riemann-Liouville definitions).

. $u(x,t)$ is the wave function.

. ∇^2 represents the Laplacian, describing spatial changes.

. f(x,t) is an external source term.

The fractional derivative order α (with $0 < \alpha \leq 2$) introduces non-locality, allowing the equation to account for the system's history.

Numerical Solution Techniques

Analytical solutions for fractional wave equations are often complex, making numerical approaches essential. Key numerical methods include, (see [135]-[136]-[137]):

1. **Finite Difference Method (FDM):**
    - This method approximates derivatives by discretizing time and space. The fractional time derivative can be approximated using schemes like the Grünwald-Letnikov or L1, which sum contributions from previous time steps, reflecting the system's memory.
2. **Finite Element Method (FEM):**

- FEM is particularly useful for problems involving irregular domains or complex boundary conditions. The domain is divided into elements, and the fractional equation is solved in its weak form. This method offers flexibility in handling diverse geometries and varying material properties.

3. **Spectral Methods:**
   - Spectral methods represent the solution as a series expansion, such as Fourier or Chebyshev series. These methods are efficient for smooth problems and can achieve high accuracy with fewer grid points.
4. **Time-stepping Schemes:**
   - Special schemes, like the L1 method, are needed to handle fractional derivatives. These schemes ensure that the numerical solution respects the non-local properties of the fractional derivative.

Applications

Fractional wave equations are applied in many fields, including, (see [135]-[136]-[137]):

- **Geophysics:** Modeling seismic waves in complex media.
- **Engineering:** Studying viscoelastic materials that exhibit both elastic and viscous behavior.
- **Medicine:** Modeling diffusion in biological tissues, where anomalous diffusion occurs.

Stability and Convergence

Ensuring numerical stability and convergence is critical. Stability ensures the solution remains bounded, while convergence guarantees that the numerical solution approximates the true solution as the mesh is refined. The choice of method, time step, and grid resolution are important factors in achieving stability and convergence (see [135]-[136]-[137]).

4.3 Finite Difference Method for Fractional Wave Equations

1. Discretizing Time and Space

The Finite Difference Method (FDM) begins by dividing the continuous problem domain into a grid. In a one-dimensional setting, the spatial domain is split into small segments of length Δx, and the time domain into intervals of length Δt. The wave function $u(x,t)$ is then approximated at discrete grid points, $u_i^n = u(x_i, t_n)$ where i and n index the space and time steps, respectively (see [135]-[138]-[139]).

2. Spatial Derivative Approximation

The second spatial derivative, which is often present in wave equations, is approximated using central differences (see [135]-[138]-[139]):

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x=x_i} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

This formula uses the values of the wave function at neighboring grid points to estimate the curvature at a given point.

3. Fractional Time Derivative Approximation

The fractional derivative in time, which is the hallmark of fractional wave equations, requires special treatment. Two common approximations are (see [135]-[138]-[139]):

4. Grünwald-Letnikov Approximation

$$D_t^{\alpha} u(x, t_n) \approx \frac{1}{(\Delta t)^{\alpha}} \sum_{k=0}^{n} \omega_k^{\alpha} u(x, t_{n-k}),$$

where the weights ω_k^{α} are calculated as:

$$\omega_k^{\alpha} = (-1)^k \binom{\alpha}{k}$$

This method sums contributions from all previous time steps, reflecting the fractional derivative's non-local nature (see [135]-[138]-[139]).

L1 Scheme:

$$D_t^{\alpha} u(x, t_n) \approx \frac{1}{(\Delta t)^{\alpha}} \sum_{k=0}^{n-1} b_k^{\alpha} \big(u(x, t_{n-k}) - u(x, t_{n-k-1}) \big)$$

where the coefficients b_k^{α} are defined as $(k+1)^{\alpha} - k^{\alpha}$. The L1 scheme is a popular choice due to its balance between computational efficiency and accuracy (see [135]-[138]-[139]).

5. Formulating the Difference Equation

By combining the spatial derivative approximation with the fractional time derivative, you form a difference equation that represents the fractional wave equation at each grid point:

$$\frac{1}{(\Delta t)^{\alpha}} \sum_{k=0}^{n} \omega_k^{\alpha} u_i^{n-k} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + f_i^n.$$

This equation can then be solved iteratively, moving step by step through time, using initial conditions and appropriate boundary conditions (see [135]-[138]-[139]).

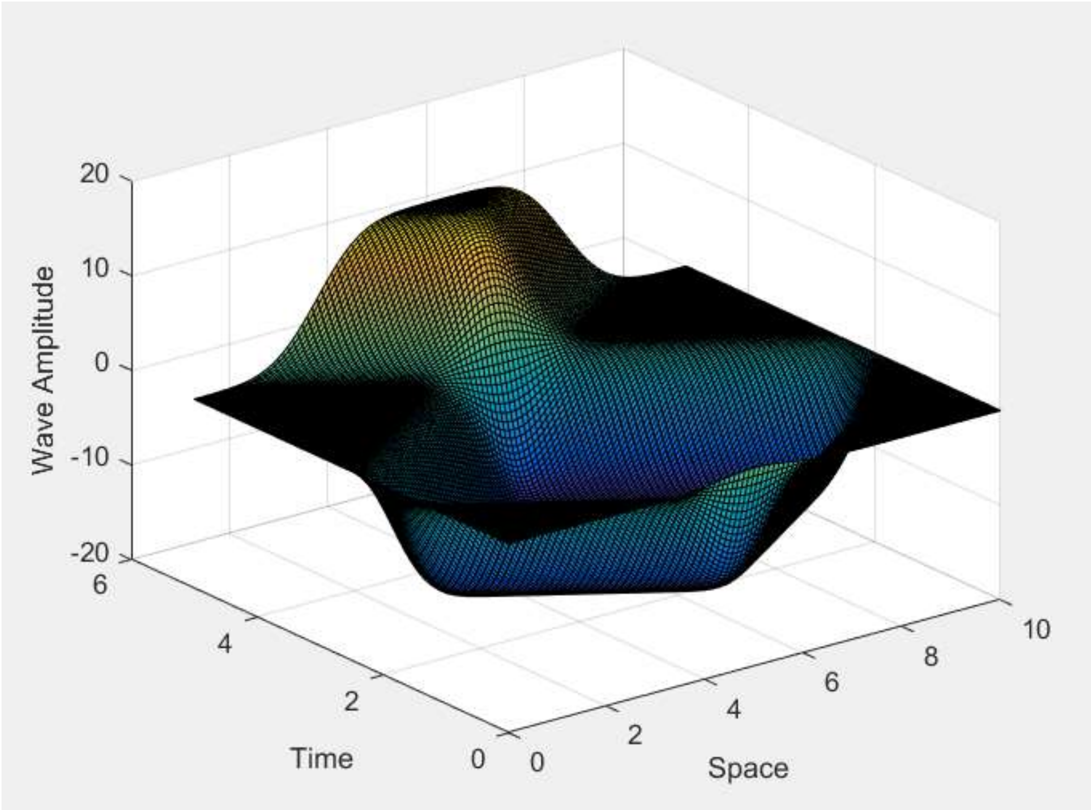
6. Ensuring Stability and Convergence

In numerical methods, stability refers to the algorithm's ability to produce bounded solutions over time, while convergence ensures that the numerical solution approximates the true solution as the grid is refined. When working with fractional wave equations, selecting appropriate time step Δt and space step Δx is crucial for maintaining stability and ensuring that the solution converges (see [135]-[138]-[139]).

7. Applications

The FDM is used to solve fractional wave equations in various fields (see [135]-[138]-[139]):

- **Geophysics:** For modeling seismic wave behavior in complex subsurface environments.
- **Material Science:** To understand wave propagation in materials with both elastic and viscous properties.
- **Medical Physics:** In modeling diffusion processes in tissues, where anomalous diffusion is observed.



4.4 Finite Element Method for Fractional Wave Equations

The Finite Element Method (FEM) is a widely-used numerical technique for solving partial differential equations (PDEs), including those involving fractional derivatives. Fractional wave equations describe wave phenomena with non-integer order derivatives, capturing complex behaviors like anomalous diffusion and wave propagation in heterogeneous media (see [140]-[141]-[142]).

1. Formulation of the Problem

- **Fractional Wave Equation**: Typically, a fractional wave equation can be expressed as:

$$D_t^{\alpha} u(x,t) - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)$$

where D_t^{α} denotes the Caputo fractional derivative of order α, (where $0 < \alpha \leq 2$), $u(x,t)$ is the wave function, c is the wave speed, and $f(x,t)$ is a source term (see [140]-[141]-[142]).

- **Initial and Boundary Conditions**: The problem is defined over a domain Ω with specified initial conditions for the wave and its time derivative, as well as boundary conditions along the domain's boundaries Γ (see [140]-[141]-[142]).
-

2. Spatial Discretization

- **Mesh Construction**: The spatial domain Ω is divided into smaller subdomains (elements), often triangles or quadrilaterals in 2D or tetrahedra in 3D, forming a mesh.
- **Elemental Approximation**: Within each element, the solution u(x,t) is approximated by a combination of basis functions, which are usually polynomials.

3. Weak Form of the Equation

- The fractional wave equation is transformed into its weak form by multiplying by a test function v(x) and integrating over the domain:

$$\int_{\Omega} v(x)\left(D_t^{\alpha}u(x,t) - c^2\frac{\partial^2 u(x,t)}{\partial x^2} - f(x,t)\right)dx = 0$$

- Integration by parts is applied to the spatial derivative term to reduce its order, facilitating numerical implementation (see [140]-[141]-[142]).

4. Time Discretization

- **Approximating Fractional Derivatives**: The fractional time derivative is discretized using methods like the Grünwald-Letnikov or Caputo schemes. The time interval is divided into discrete steps, and the fractional derivative is approximated as a sum over previous time steps:

$$D_t^{\alpha}u(x,t) \approx \sum_{k=0}^{n} \omega_k^{\alpha}\, u(x, t_{n-k}),$$

- where $\omega_k^{\alpha}$are the coefficients associated with the chosen scheme (see [140]-[141]-[142]).

5. Element Assembly

- **Global Stiffness Matrix**: The weak form leads to the assembly of a global stiffness matrix representing the discretized PDE. Each mesh element contributes a local stiffness matrix, which is aggregated to form the global system (see [140]-[141]-[142]).
- **Load Vector**: The load vector, representing the source term and boundary conditions, is constructed in a similar manner, also incorporating the effects of the fractional derivative (see [140]-[141]-[142]).

6. Solving the System of Equations

- The resulting linear or nonlinear system of equations at each time step can be represented as:

$$KU^n = F^n$$

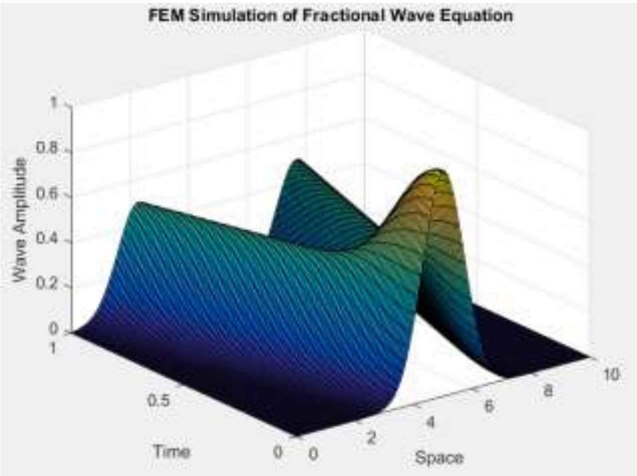
where K is the global stiffness matrix, U^n is the vector of the wave function's nodal values at time step n, and F^n is the global load vector (see [140]-[141]-[142]).

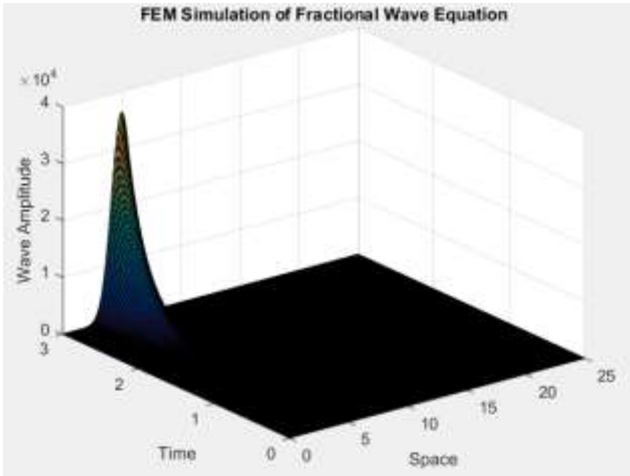
7. Time Integration and Solution

- **Iterative Time Stepping**: The solution progresses through time by solving the system at each time step, starting from the initial conditions. Numerical solvers like LU decomposition or iterative methods (e.g., Conjugate Gradient) are employed to solve the system (see [140]-[141]-[142]).
- **Ensuring Stability and Convergence**: The time step size Δt and mesh resolution must be carefully chosen to ensure numerical stability and accuracy (see [140]-[141]-[142]).

8. Applications and Use Cases

- **Geophysics**: FEM is used to model wave propagation in complex geological media (see [140]-[141]-[142]).
- **Material Science**: Studying wave behavior in materials exhibiting memory effects or fractal structures (see [140]-[141]-[142]).
- **Biomedical Engineering**: Analyzing wave propagation in biological tissues with anomalous diffusion characteristics (see [140]-[141]-[142]).





4.5 Shallow water waves: Overview and simulation

1. Introduction to Shallow Water Waves

Shallow water waves are a type of wave that occurs in bodies of water where the depth is small relative to the wavelength. These waves are typically characterized by the following properties:

- **Depth Condition**: The water depth h is much smaller than the wavelength λ such that $h \ll \lambda$.This condition means that the wave is influenced by the bottom surface of the water body.
- **Wave Speed**: The speed c of shallow water waves depends on the depth of the water and is given by $c = \sqrt{gh}$, where g is the acceleration due to gravity.

Shallow water waves are important in various contexts, such as in coastal engineering, where they can model tides, tsunamis, and other wave phenomena in oceans and seas.

2. **Governing Equations**

The primary equations used to describe shallow water waves are the Shallow Water Equations (SWEs)**.** These are a set of hyperbolic partial differential equations derived from the conservation of mass and momentum.

3. Continuity Equation:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

where:

- $\eta(x, t)$ is the free surface elevation,
- $u(x, t)$ is the velocity of the water in the horizontal direction,
- $h(x, t) = H + \eta(x, t)$ is the total water depth, with H being the mean water depth.

4. Momentum Equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0.$$

These equations are used to model the wave propagation in shallow water bodies, accounting for the variations in water height and velocity over time.

4.6 Numerical Solution of Shallow Water Wave Equations Using the Finite Difference Method (FDM)

1. Introduction to Shallow Water Wave Equations

Shallow water wave equations describe the behavior of waves in environments where the depth of the water is relatively small compared to the wavelength of the waves. These equations are derived from the conservation of mass and momentum and are widely used to simulate natural phenomena such as tides, storm surges, and tsunamis.

The equations typically consist of (see [143]-[144]-[145]-[146]-[147]-[148]):

- **Continuity Equation**: Represents the conservation of mass by linking the change in water height with the flow velocity.
- **Momentum Equation**: Describes the movement of water by balancing forces like pressure gradients and gravitational acceleration.

2. Finite Difference Method (FDM) for Shallow Water Waves

The Finite Difference Method (FDM) is a numerical technique that approximates the solutions of differential equations by discretizing the spatial and temporal domains into a grid. For shallow water wave equations, FDM is used to approximate the changes in water height and velocity over time (see [143]-[144]-[145]-[146]-[147]-[148]).

2.1 Discretization

- **Spatial Discretization**: The domain is divided into small intervals Δx, where the water surface elevation $\eta(x, t)$ and velocity $u(x, t)$ are computed at discrete points.
- **Temporal Discretization**: Time is divided into intervals Δt, with values of η and u updated at each step.

2.2 FDM Scheme

The finite difference approximations for the shallow water wave equations are applied as follows (see [143]-[144]-[145]-[146]-[147]-[148]):

Continuity Equation:

$$\frac{\partial \eta}{\partial t} \approx \frac{\eta_i^{n+1} - \eta_i^n}{\Delta t}$$

$$\frac{\partial}{\partial x}(hu) \approx \frac{(hu)_{i+1}^n - (hu)_{i-1}^n}{2\Delta x}$$

Momentum Equation:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$u\frac{\partial u}{\partial x} \approx u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

$$g\frac{\partial \eta}{\partial x} \approx g\frac{\eta_{i+1}^n - \eta_{i-1}^n}{2\Delta x}$$

In this scheme, i represents the spatial grid point, and n represents the time step. The variables η and u are iteratively updated to simulate the wave's evolution over time (see [143]-[144]-[145]-[146]-[147]-[148]).

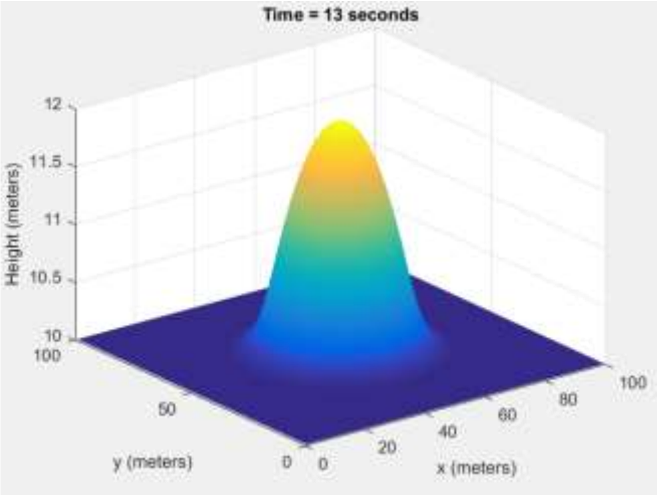


Figure: Shallow water waves when $t = 13$s

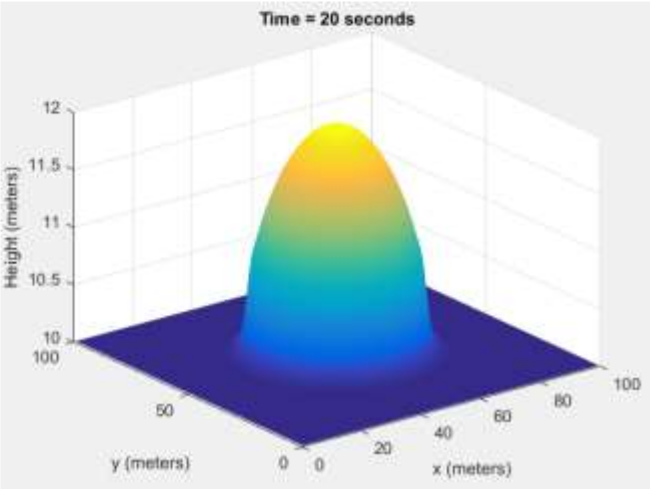


Figure: Shallow water waves when $t = 20$s

4.7 Caputo fractional operator on shallow water wave equations

The shallow water wave equations can be adapted to incorporate fractional calculus, providing a more nuanced model for wave behavior that can better capture phenomena such as anomalous diffusion and complex dispersion effects [149]-[150]-[151]-[152].

1 Fractional Shallow Water Wave Equations

When fractional calculus is applied to the shallow water equations, the modified equations become [149]-[150]-[151]-[152]:

Fractional Continuity Equation:

$$ {}^{C}D_t^{\alpha} h + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 $$

Fractional Momentum Equations:

$$ {}^{C}D_t^{\alpha} u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x} = 0 $$
$$ {}^{C}D_t^{\alpha} v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h}{\partial y} = 0 $$

In these equations, the Caputo fractional derivative ${}^{C}D_t^{\alpha}$ introduces memory effects and hereditary properties into the wave dynamics (see [149]-[150]-[151]-[152]).

4.8 Numerical model of Caputo fractional shallow water wave equations

Numerical Solution Approach:

To solve these fractional equations numerically, the following steps are generally involved [149]-[150]-[151]-[152]:

Temporal Discretization

The Caputo fractional derivative is approximated using numerical techniques. A common approach is to use a finite difference method for approximating fractional derivatives:

$$ {}^{c}D_t^{\alpha} f(t) \approx \frac{1}{\Gamma(n-\alpha)} \sum_{i=0}^{k-1} \frac{f^{(n)}(t_i)}{(t-t_i)^{\alpha-n+1}} \Delta t $$

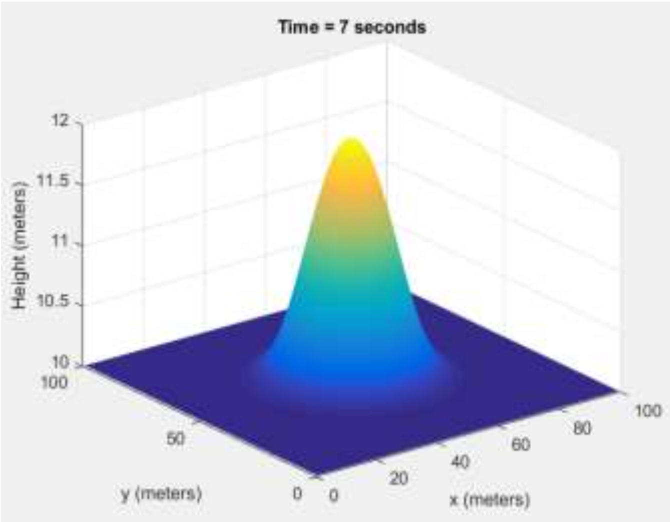
where t_i represents previous time steps and Δt is the time step size [149]-[150]-[151]-[152].

Spatial Discretization

The spatial domain is discretized using a grid, and spatial derivatives are approximated using finite difference methods. For example, central differences can be used to approximate spatial derivatives:

$$ \frac{\partial h}{\partial x} \approx \frac{h_{i+1,j} - h_{i-1,j}}{2\Delta x} $$

where $h_{i,j}$ denotes the height at grid point (i, j), and Δx is the grid spacing in the x-direction [149]-[150]-[151]-[152].



$$ {}^{C}D_{t=7}^{0.8}u + \frac{\partial(\mathrm{hu})}{\partial \mathrm{x}} + \frac{\partial(\mathrm{hv})}{\partial \mathrm{y}} = 0 $$

h - dt * (u .* hx + v .* hy)
u = u - dt * (u .* ux + v .* uy + g * hx).
v = v - dt * (u .* vx + v .* vy + g * hy).

V. CONCLUSION

The aim of this paper was to first introduce fractional differential operators and then to use the available information to discuss wave types, wave structures, and their mathematical interpretations in physics and engineering. Subsequently, by applying fractional derivative and integral operators, we sought to provide a different perspective by solving wave equations mathematically and using some numerical methods. Finally, we conducted simulations of the results obtained from these numerical values and used the fractional Caputo-type differential operators in shallow water wave theory to analyze the time-dependent behavior of a shallow water wave, the changes in wavelength over time, and their analyses based on the data we had. This study provided a different perspective to our research, especially in the field of mathematical modeling, which has become popular in recent times and is being studied by many mathematicians and engineers, through the use of various sources and academic websites.

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