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Research Article

Some Results of Ideal Convergence of Double Sequences in Topological Spaces

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Abstract **–** Here, we provide statistical $\mathfrak{A}^{\mathcal{J}_2}$-strong convergence and $\mathfrak{A}_{\mathcal{T}}^{\mathcal{J}_2}$-strong convergence, which, via a certain class of special functions, expand $\mathfrak{A}^{\mathcal{J}_2}$-strong convergence in Hausdorff topological spaces. Furthermore, we give a characterization of $\mathfrak{A}^{\mathcal{J}_2}$-statistical convergence and draw linkages between $\mathfrak{A}^{\mathcal{J}_2}$-statistical convergence and $\mathfrak{A}_{\mathcal{T}}^{\mathcal{J}_2}$-strong convergence.

Keywords – Ideal Convergence, Strong Convergence, Statistical Convergence, Topological Space.

I. INTRODUCTION

Summability theory currently holds a fascinating position within the context of topological spaces. This theory enables assigning limits to scalar-valued or vector-valued sequences, even if these sequences do not naturally converge [2]. Researchers have explored summability theory in topological spaces, assuming the space possesses a group or vector structure. However, certain summability methods, such as statistical convergence [6, 8, 9, 11] and A-statistical convergence [12] (see [3, 16, 21]), do not necessitate a linear structure within the topological space. More recent contributions to this field can be found in [10, 13, 17], where numerous references are cited. For metric spaces, Kostyrko et al. initially proposed the idea of $\mathcal{J}$-convergence in 2000-2001 [14], and this concept was further examined by Lahiri and Das in 2005 [15] in the context of topological spaces.

The statistical convergence framework may be investigated in any Hausdorff topological space as it is compatible with the topological structure, i.e., it can be described by components of the topological base. However, this approach fails when addressing strong convergence, as its definition heavily depends on the metric function. We investigate strong convergence in a topological space in this research by studying a class of pre-metrics with certain qualities that are intimately associated with the topological base elements.

A class of pre-metrics on arbitrary Hausdorff topological spaces, exhibiting several characteristics akin to the topological base, has been utilized to initiate the study of strong convergence, specifically $\mathfrak{A}_{\mathcal{T}}$-strong convergence, as discussed in [22]. In this manuscript, following the framework of [22], we have explored strong convergence through the lens of $\mathcal{J}$-convergence theory. Our results indicate that our findings are more robust compared to those of classical strong convergence.

Connor [4], Khan and Orhan [11] established the relation between $\mathfrak{A}$-statistical convergence and $\mathfrak{A}$-strong convergence. Also, in the field of $\mathfrak{A}^{\mathcal{J}}$-statistical convergence Savaş, Das and Dutta [18-19] generalized its relation with $\mathfrak{A}^{\mathcal{J}}$-summability.
Now we recall some well-known notions from literature. According to Maio et. al.[16] "Let $\mathfrak{A} = (b_{nk})$ is a regular summability matrix that is non-negative. If $\{\mathbb{t}_k\}_{k\in\mathbb{N}}$ is a sequence in a Hausdorff topological space $\mathfrak{H}$ such that for every open set O that contains a

$$\lim_n \sum_{k:\mathbb{t}_k \notin O} b_{nk} = 0$$

then the sequence $\{\mathbb{t}_k\}_{k\in\mathbb{N}}$ is called $\mathfrak{A}$-statistically convergent to $a \in \mathfrak{H}$ ".

In 1988-1989, J. S. Connor [4, 5] defined $\mathfrak{A}$-strongly convergence as "For a $\mathfrak{A} = (b_{nk})$, a sequence $\{\mathbb{t}_k\}_{k\in\mathbb{N}}$ in a metric space $(\mathfrak{H}, d)$ is called to be $\mathfrak{A}$-strongly convergent to $\alpha \in X$ if

$$\lim_n \sum_k d(\mathbb{t}_k, \alpha) b_{nk} = 0\text{"}$$

In 2012-2013, Savaş et. al. [18, 19] defined $\mathfrak{A}^{\mathcal{J}}$-statistically convergence as "For a $\mathfrak{A} = (b_{nk})$, sequence $\{\mathbb{t}_k\}_{k\in\mathbb{N}}$ is said to be $\mathfrak{A}^{\mathcal{J}}$-statistically convergent to α if for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N}: \sum_{k\in\mathfrak{C}(\varepsilon)} b_{nk} \geq \delta \right\} \in \mathcal{J}$$

where $\mathfrak{C}(\varepsilon) = \{k \in \mathbb{N}: |\mathbb{t}_k - \alpha| \geq \varepsilon\}$".

In this study, we delve into the intriguing realm of topological spaces by introducing the concepts of statistical $\mathfrak{A}^{\mathcal{J}_2}$-strong convergence and $\mathfrak{A}^{\mathcal{J}_2}$-strong convergence. These new forms of convergence extend the idea of $\mathfrak{A}^{\mathcal{J}_2}$-strong convergence within Hausdorff topological spaces through the use of a specific class of special functions. Our exploration uncovers fascinating connections between $\mathfrak{A}^{\mathcal{J}_2}$-statistical convergence and $\mathfrak{A}^{\mathcal{J}_2}$-strong convergence, offering a fresh perspective on their interplay. Additionally, we provide a comprehensive characterization of $\mathfrak{A}^{\mathcal{J}_2}$-statistical convergence, paving the way for a deeper understanding of summability in topological spaces.

II. MAIN RESULTS

The notion of strong convergence cannot be analyzed in arbitrary topological spaces due to its reliance on metric functions. Consequently, the well-established link between strong convergence and statistical convergence is not easily applicable to topological spaces. In this section, we introduce $\mathfrak{A}^{\mathcal{J}_2}$-statistical convergence within a topological space and define $\mathfrak{A}^{\mathcal{J}_2}_{\mathcal{T}}$-strong convergence using a specific function defined on the topological space. We explore the relationships between these types of convergence. Throughout $\lfloor NN - R - S \rfloor$ will represent the class of all non-negative regular summability matrices.

Definition 2.1. Let $\mathfrak{A} = (b_{r,s,\mathrm{i,j}}) \in \lfloor NN - R - S \rfloor$. A sequence $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ in a Hausdorff topological space $\mathfrak{H}$ is called to be $\mathfrak{A}^{\mathcal{J}_2}$-statistically convergent to $\alpha \in \mathfrak{H}$ if

$$\left\{(r,s)\in\mathbb{N}^2:\sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}} b_{r,s,\mathrm{i,j}}\geq \varsigma\right\}\in \mathcal{J}_2$$

for any $\varsigma > 0$ and for any open set O containing α.

It is straightforward to observe that, in lie of open sets, the elements of the topology's base can be employed to define $\mathfrak{A}^{\mathcal{J}_2}$-statistically convergence.

Utilizing the concept of pre-metric [1] and specific properties in a topological space, we define $\mathfrak{A}^{\mathcal{J}_2}$-strong convergence in a topological space. Let $(\mathfrak{H},\tau)$ be a Hausdorff topological with $\alpha \in \mathfrak{H}$. Define $B_{\mathcal{E}}(\alpha,\lambda):=\{\beta\in\mathfrak{H}:\mathcal{E}(\beta,\alpha)<\lambda\}$ and designate $\mathcal{B}_\alpha$ as the family of elements in the base of τ containing α. Additionally, we define $\mathcal{L}(\mathfrak{H})$ as the collection of functions $\mathcal{E}:\mathfrak{H}\times\mathfrak{H}\to[0,\infty)$ that meet the following requirement: "There exists $\mathrm{O}_\lambda\in\mathcal{B}_\alpha$ such that $\mathrm{O}_\lambda\subset B_{\mathcal{E}}(\alpha,\lambda)$ for any $\lambda>0$ and $\alpha\in\mathfrak{H}$" [21]. It is not necessary for the topology to be pre-metrizable, even if every function from $\mathcal{L}(\mathfrak{H})$ is a pre-metric (for instance, [1]) on $\mathfrak{H}$. These pre-metrics are partly compatible with the topology due to their satisfaction of this specific condition.

Definition 2.2. Let $\mathfrak{A}=\left(b_{r,s,\mathrm{i,j}}\right)\in\lfloor NN-R-S\rfloor$, $\mathcal{T}\subset\mathcal{L}(\mathfrak{H})$, and let $(\mathfrak{H},\tau)$ be a Hausdorff topological space. A sequence $\left\{\mathbb{t}_{\mathrm{i,j}}\right\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ in $\mathfrak{H}$ is called to be statistical $\mathfrak{A}_{\mathcal{T}}$-strongly convergent to $\alpha\in\mathfrak{H}$ if for any $\mathcal{E}\in\mathcal{T}$, the sequence $\left\{\sum_{\mathrm{i,j}}\mathcal{E}\left(\mathbb{t}_{\mathrm{i,j}},\alpha\right)b_{r,s,\mathrm{i,j}}\right\}_{(r,s)\in\mathbb{N}^2}$ is statistically convergent to zero, provided the series is convergent for each $(r,s)\in\mathbb{N}^2$.

Definition 2.3. Let $\mathfrak{A}=\left(b_{r,s,\mathrm{i,j}}\right)\in\lfloor NN-R-S\rfloor$, $\mathcal{T}\subset\mathcal{L}(\mathfrak{H})$ and let $(\mathfrak{H},\tau)$ be a Hausdorff topological space. A sequence $\left\{\mathbb{t}_{\mathrm{i,j}}\right\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ in $\mathfrak{H}$ is called to be $\mathfrak{A}_{\mathcal{T}}^{\mathcal{J}_2}$-strongly convergent to $\alpha\in\mathfrak{H}$ provided that for any $\Upsilon>0$ and for any $\mathcal{E}\in\mathcal{T}$

$$\left\{(r,s)\in\mathbb{N}^2:\sum_{\mathrm{i,j}}\mathcal{E}\left(\mathbb{t}_{\mathrm{i,j}},\alpha\right)b_{r,s,\mathrm{i,j}}\geq\Upsilon\right\}\in\mathcal{J}_2$$

provided the series is convergent for each $(r,s)\in\mathbb{N}^2$.

For $\mathcal{J}_2=\mathcal{J}_d^2$, the ideal consisting of all the subsets of $\mathbb{N}^2$ with natural density zero, the aforementioned convergences coincide. Here, we provide an example of an $\mathfrak{A}_{\mathcal{T}}^{\mathcal{J}_2}$-strongly convergent sequence in a non-metrizable Hausdorff space.

Example 2.1. Consider the topological space $(\mathbb{R}^2,\tau_L)$, where τ_L be the lower limit topology. It is a non-metrizable Hausdorff space [20]. The sequence $\left\{\mathbb{t}_{\mathrm{i,j}}\right\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ defined by

$$\mathbb{t}_{\mathrm{i,j}}=\begin{cases}0, & \text{if } \mathrm{i,j} \text{ are even,}\\ 1, & \text{otherwise}\end{cases}$$

Is what we will consider. Next, let us examine the set of functions $\mathcal{T}=\{\mathcal{E}_r\}_{r\geq 0}$ that are defined for any $r\geq 0$ for which

$$\mathcal{E}_r(a,b)=\begin{cases}a-b, & a\geq b\\ r, & a<b.\end{cases}$$

Then $\mathcal{E}_r \in \mathcal{T}$. Let $\mathcal{I}_2$ be a non-trivial admissible ideal of $\mathbb{N}^2$ such that $\mathcal{I}_2 = \mathcal{I}_{\text{fin}}^2$, (the ideal containing all finite subsets of $\mathbb{N}^2$). Consider an infinite set $\mathcal{D} = \{q_1 < q_2 < q_3 < \cdots\}$ from $\mathcal{I}_2$ and take the infinite matrix $\mathfrak{A} = (b_{r,s,\mathrm{i,j}})$ be given by,

$$b_{r,s,\mathrm{i,j}} = \begin{cases} 1, & \text{if } r,s = q_i, \quad \mathrm{i,j} = 2q_i \text{ for some } i \in \mathbb{N}, \\ 1, & \text{if } r,s \neq q_i, \text{ for any } i, \quad \mathrm{i} = 2r+1, \mathrm{j} = 2s+1, \\ 0, & \text{otherwise.} \end{cases}$$

For any $\curlyvee > 0$ and $\alpha \in \mathfrak{H}$ we consider $0_{\curlyvee} = [\alpha, \alpha + \curlyvee) \in \mathcal{B}_\alpha$ then for all $\beta \in 0_{\curlyvee}$ we have $\mathcal{E}(\beta, \alpha) = \beta - \alpha < \curlyvee$. So $\mathcal{T} \subset \mathcal{L}(\mathfrak{H})$ and $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j}) \in \mathbb{N}^2}$ is not convergent in the corresponding topology. But, we have

$$\mathcal{E}_r(\mathbb{t}_{\mathrm{i,j}}, 1) = \begin{cases} r, & \mathrm{i,j} \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

For any $r \geq 0$ and $\varsigma > 0$ we get that

$$\sum_{\mathrm{i,j}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, 1) b_{r,s,\mathrm{i,j}} = \sum_{(\mathrm{i,j}) \in 2\mathbb{N}^2} b_{r,s,\mathrm{i,j}} = \sum_{(\mathrm{i,j}) \in 2\mathbb{N}^2} r \geq \varsigma,$$

for $r, s = q_i$. So, for any $\varsigma > 0$

$$\left\{ (r,s) \in \mathbb{N}^2 : \sum_{\mathrm{i,j}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, 1) b_{r,s,\mathrm{i,j}} \geq \varsigma \right\} = \mathcal{D} \in \mathcal{I}_2,$$

demonstrates that the sequence $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j}) \in \mathbb{N}^2}$ is $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergent to 1 but not $\mathfrak{A}_{\mathcal{T}}$-strong convergent to 1. Thus, $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strong convergence is stronger than $\mathfrak{A}_{\mathcal{T}}$-strong convergence.

In particular, if we choose $\mathcal{I}_2 \neq \mathcal{I}_{\text{fin}}^2$ and $\mathcal{I}_2 \neq \mathcal{I}_d^2$ and the infinite set $\mathcal{D} \in \mathcal{I}_2 \setminus \mathcal{I}_d^2$ then the sequence $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j}) \in \mathbb{N}^2}$ is $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergent to 1 but not statistical $\mathfrak{A}_{\mathcal{T}}$-strongly convergent to 1.

The set of functions $\mathcal{E}: \mathfrak{H} \times \mathfrak{H} \to [0, \infty)$ that meet the requirement: "For all $\alpha \in \mathfrak{H}$ and for all $B \in \mathcal{B}_\alpha$ there exists $\mathfrak{M} > 0$ such that for every $\beta \notin B, \mathcal{E}(\beta, \alpha) > \mathfrak{M}$" is denoted as $\mathfrak{J}(\mathfrak{H})$ [21].

Theorem 2.1. Let $\mathfrak{A} = (b_{r,s,\mathrm{i,j}}) \in \lfloor NN - R - S \rfloor$, $\mathcal{T} \subset \mathcal{L}(\mathfrak{H})$ and let $\mathfrak{H}$ be a Hausdorff topological space and. So, we have

(i) $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergence with $\mathcal{T} \cap \mathfrak{J}(\mathfrak{H}) \neq \emptyset$ implies $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergence.

(ii) $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergence with the condition

$$\sup_{\mathcal{E} \in \mathcal{T}} \sup_{\mathrm{i,j}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, \alpha) < \infty, \tag{1}$$

follows $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergence.

Proof. (i) Let a sequence $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j}) \in \mathbb{N}^2}$ be $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergent to α in $\mathfrak{H}$. Let $B \in \mathcal{B}_\alpha$ and $\mathcal{E} \in \mathcal{T} \cap \mathfrak{J}(\mathfrak{H})$. So, there exists $\mathfrak{M} > 0$ such that for all $\beta \notin B, \mathcal{E}(\beta, \alpha) > \mathfrak{M}$. From non-negativity of $\mathcal{E}$ we get

$$\sum_{i,j}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}=\sum_{(i,j):\mathbb{t}_{i,j}\in B}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}+\sum_{(i,j):\mathbb{t}_{i,j}\notin B}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}\geq\sum_{(i,j):\mathbb{t}_{i,j}\notin B}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}$$
$$\geq\mathfrak{M}\sum_{(i,j):\mathbb{t}_{i,j}\notin B}b_{r,s,i,j}.$$

Let $\varsigma>0$ be given. Therefore

$$\left\{(r,s)\in\mathbb{N}^2:\sum_{(i,j):\mathbb{t}_{i,j}\notin B}b_{r,s,i,j}\geq\varsigma\right\}\subseteq\left\{(r,s)\in\mathbb{N}^2:\sum_{i,j}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}\geq\varsigma\cdot\mathfrak{M}\right\}.$$

As the sequence $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergent to α in $\mathfrak{H}$ then the right side set belongs to $\mathcal{I}_2$ and so, we have

$$\left\{(r,s)\in\mathbb{N}^2:\sum_{(i,j):\mathbb{t}_{i,j}\notin B}b_{r,s,i,j}\geq\varsigma\right\}\in\mathcal{I}_2.$$

As B is arbitrary, the sequence $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to α. Hence the proof of (i) is completed.

(ii) Let us suppose $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to $\alpha\in X$ and satisfies condition (1). Let $\mathcal{E}\in\mathcal{T}$. Then, for any $\curlyvee>0$ there exists $B\in\mathcal{B}_\alpha$ such that for all $\beta\in B,\mathcal{E}(\beta,\alpha)<\curlyvee$. Therefore, the non-negativity of $\mathcal{E}$ implies

$$\sum_{i,j}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}=\sum_{(i,j):\mathbb{t}_{i,j}\in B}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}+\sum_{(i,j):\mathbb{t}_{i,j}\notin B}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}$$
$$\leq\curlyvee\sum_{(i,j):\mathbb{t}_{i,j}\in B}b_{r,s,i,j}+\sup_{T\in\mathcal{T}}\sup_{i,j}\mathcal{E}(\mathbb{t}_{i,j},\alpha)\sum_{(i,j):\mathbb{t}_{i,j}\notin B}b_{r,s,i,j}\leq\curlyvee+\mathfrak{M}\sum_{(i,j):\mathbb{t}_{i,j}\notin B}b_{r,s,i,j},$$

where $\mathfrak{M}=\sup_{T\in\mathcal{T}}\sup_{i,j}\mathcal{E}(\mathbb{t}_{i,j},\alpha)$. For a given $\sigma>0$ select $\curlyvee>0$ such that $\curlyvee<\sigma$. Then

$$\left\{(r,s)\in\mathbb{N}^2:\sum_{i,j}\mathcal{E}(\mathbb{t}_{i,j},\alpha)b_{r,s,i,j}\geq\curlyvee\right\}\subseteq\left\{(r,s)\in\mathbb{N}^2:\sum_{(i,j):\mathbb{t}_{i,j}\notin B}b_{r,s,i,j}\geq\frac{\sigma-\curlyvee}{\mathfrak{M}}\right\}.$$

Since $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to $\alpha\in\mathfrak{H}$, the set on the right side belongs to $\mathcal{I}_2$ and consequently this implies that $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergent to α.

Remark 2.1. In Example 2.1, we provided an example of a sequence that is $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strongly convergent. We note that $\mathfrak{J}(\mathfrak{H})$ contains the family $\mathcal{T}$ as a subset. For $\mathcal{E}_r\in\mathcal{T},\alpha\in\mathbb{R}$ and $O\in\mathcal{B}_\alpha$ we have O is in the most general form of $[\alpha,\gamma)$. Then, for any $\beta\notin U,\mathcal{E}_r(\beta,\alpha)=\beta-\alpha$ when $\beta>\alpha$; otherwise $\mathcal{E}_r(\beta,\alpha)=r$. In addition $\mathcal{E}(\gamma,\alpha)=\gamma-\alpha$ and since $\beta>\alpha$ implies $\beta>\gamma$, it follows that $\mathcal{E}_r(\beta,\alpha)>\mathcal{E}(\gamma,\alpha)$. Therefore $\mathcal{E}_r(\beta,\alpha)>\min\left\{\frac{r}{2},\mathcal{E}(\gamma,\alpha)\right\}$. Hence, by Theorem 2.1, the sequence given in Example 2.1 converges $\mathfrak{A}^{\mathcal{I}_2}$-statistically to 1 with respect to the specified ideal.

On the other hand, in the previous example, since the given sequence is $\mathfrak{A}^{\mathcal{J}_2}$-statistically convergent to 1, we can define a family of functions $\mathcal{T} = \{\mathcal{E}_r\}_{r\geq 0}$ for any $r \geq 0$ as

$$\mathcal{E}_r(a,b) = \begin{cases} a-b, & a \geq b \\ \dfrac{1}{r+1}, & a < b \end{cases}$$

This family satisfies condition (1), namely $\sup_{\mathcal{E}\in\mathcal{T}} \sup_{\mathrm{i,j}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, 1) = 1 < \infty$ and also $\mathcal{T} \subset \mathcal{L}(\mathfrak{H}) \cap \mathfrak{J}(\mathfrak{H})$. Therefore, it supports part (ii) of Theorem 2.1.

Let $d \in \mathcal{L}(\mathfrak{H}) \cap \mathfrak{J}(\mathfrak{H})$ and let $(\mathfrak{H}, d)$ be a metric space. Any bounded sequence satisfying (1) if $\mathcal{T} = \{d\}$ is obtained, and for $\mathcal{J}_2 = \mathcal{J}_{\mathrm{fin}}^2$, the ideal encompassing all finite subsets of $\mathbb{N}^2$, the following consequence is obtained.

Corollary 2.1. [4] Let $\mathfrak{A} = (b_{r,s,\mathrm{i,j}})$ be a regular summability matrix that is non-negative. When a sequence in $\mathfrak{H}$ is $\mathfrak{A}$-strongly convergent to $\alpha \in \mathfrak{H}$, it also $\mathfrak{A}$-statistically converges to α. $\mathfrak{A}$-strongly convergence and $\mathfrak{A}$-statistically convergence are equivalent for a bounded sequence in $\mathfrak{H}$.

Theorem 2.2. Let $\mathcal{T} \subset \mathcal{L}(\mathfrak{H})$, where $\mathfrak{H}$ is a Hausdorff topological space, and let $\mathfrak{A} = (b_{r,s,\mathrm{i,j}}) \in \lfloor NN - R - S \rfloor$. If a sequence $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{J}_2}$-statistically convergent to α in X and satisfies

(i) if there is a compact subset $F \subset \mathfrak{H}$ such that

$$\sup_{r,s} \sum_{\mathbb{t}_{\mathrm{i,j}} \notin F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, \alpha) b_{r,s,\mathrm{i,j}} < \Upsilon$$

exists for every $\Upsilon > 0$ and $\mathcal{E} \in \mathcal{T}$.

(ii) for any compact set $C \subset X$ and $\mathcal{E} \in \mathcal{T}$ there exists $\mathfrak{M} > 0$ such that $\sup_{\mathbb{t}_{\mathrm{i,j}}\in C} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, \alpha) < \mathfrak{M}$ then $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ is $\mathfrak{A}_{\mathcal{T}}^{\mathcal{J}_2}$-strongly convergent to α.

Proof. Consider a sequence $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ in $\mathfrak{H}$ that is $\mathfrak{A}^{\mathcal{J}_2}$-statistically convergent to α. Assume that $\Upsilon > 0$ and $\mathcal{E} \in \mathcal{T}$. Furthermore, based on the premise

$$\sup_{r,s} \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}} \notin F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, \alpha) b_{r,s,\mathrm{i,j}} < \frac{\Upsilon}{2}$$

exists for a compact set $F \subset \mathfrak{H}$. And there exists $\mathfrak{M} > 0$ such that

$$\sup_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}} \in F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}}, \alpha) < \mathfrak{M}.$$

As $\mathcal{E} \in \mathcal{L}(\mathfrak{H})$ there exists $0_\Upsilon \in \mathcal{B}_\alpha$ such that $0_\Upsilon \subset B_{\mathcal{E}}\left(\alpha, \frac{\Upsilon}{2}\right)$. Now for any positive integers r, s

$$\begin{aligned}\sum_{\mathrm{i,j}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} &= \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\in F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} + \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\notin F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} \\ &= \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\in F\cap \mathrm{O}_{\curlyvee}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} + \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\in F\setminus \mathrm{O}_{\curlyvee}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} + \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\notin F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} \\ &\le \frac{\curlyvee}{2}\sum_{\mathrm{i,j}} b_{r,s,\mathrm{i,j}} + \sup_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\in F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}_{\curlyvee}} b_{r,s,\mathrm{i,j}} + \sup_{r,s} \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\notin F} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} \\ &\le \frac{\curlyvee}{2}\sum_{\mathrm{i,j}} b_{r,s,\mathrm{i,j}} + \mathfrak{M} \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}_{\curlyvee}} b_{r,s,\mathrm{i,j}} + \frac{\curlyvee}{2}.\end{aligned}$$

As $\mathfrak{A} = (b_{r,s,\mathrm{i,j}})$ is regular summability matrix then for any given $\varsigma > 0$ choosing $\curlyvee < \sigma$

$$\left\{(r,s) \in \mathbb{N}^2 : \sum_{\mathrm{i,j}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} \ge \varsigma\right\} \subseteq \left\{(r,s) \in \mathbb{N}^2 : \sum_{(\mathrm{i,j}):\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}_{\curlyvee}} b_{r,s,\mathrm{i,j}} \ge \frac{\varsigma - \curlyvee}{\mathfrak{M}}\right\}.$$

Since the right hand side set belongs to $\mathcal{I}_2$,

$$\left\{(r,s) \in \mathbb{N}^2 : \sum_{\mathrm{i,j}} \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha) b_{r,s,\mathrm{i,j}} \ge \varsigma\right\} \in \mathcal{I}_2$$

for any $\varsigma > 0$. Hence the proof is completed.

A characterisation of $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergence is found in the final theorem.

Theorem 2.3. Suppose that $\mathfrak{H}$ is a Hausdorff topological space. For each $\mathfrak{A} = (b_{r,s,\mathrm{i,j}}) \in \lfloor NN - R - S \rfloor$, let's say that $\mathcal{T} \subset \mathcal{L}(\mathfrak{H}) \cap \mathfrak{J}(\mathfrak{H})$. Then, a sequence $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ in $\mathfrak{H}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to $\alpha \in \mathfrak{H}$ iff

$$\mathcal{I}_2 - \lim_{r,s} \sum_{\mathrm{i,j}} \frac{\mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)}{1 + \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)} b_{r,s,\mathrm{i,j}} = 0. \qquad (2)$$

Proof. Initially, let $\curlyvee > 0$ and assume that $\{\mathbb{t}_{\mathrm{i,j}}\}_{(\mathrm{i,j})\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to $\alpha \in \mathfrak{H}$. There exists $\mathrm{O}_{\curlyvee} \in \mathcal{B}_\alpha$ such that $\mathrm{O}_{\curlyvee} \subset B_{\mathcal{E}}(\alpha,\curlyvee)$ as $\mathcal{E} \in \mathcal{L}(\mathfrak{H})$. Then, we have

$$\begin{aligned}\sum_{\mathrm{i,j}} \frac{\mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)}{1 + \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)} b_{r,s,\mathrm{i,j}} &= \sum_{\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}_{\curlyvee}} \frac{\mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)}{1 + \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)} b_{r,s,\mathrm{i,j}} + \sum_{\mathbb{t}_{u,v}\in \mathrm{O}_{\curlyvee}} \frac{\mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)}{1 + \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)} b_{r,s,\mathrm{i,j}} \\ &\le \sum_{\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}_{\curlyvee}} b_{r,s,\mathrm{i,j}} + \curlyvee \sum_{\mathbb{t}_{\mathrm{i,j}}\in \mathrm{O}_{\curlyvee}} b_{r,s,\mathrm{i,j}} \le \sum_{\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}_{\curlyvee}} b_{r,s,\mathrm{i,j}} + \curlyvee.\end{aligned}$$

For each $\varsigma > 0$, select $\curlyvee > 0$ such that $\curlyvee < \varsigma$. Then

$$\left\{(r,s) \in \mathbb{N}^2 : \sum_{\mathrm{i,j}} \frac{\mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)}{1 + \mathcal{E}(\mathbb{t}_{\mathrm{i,j}},\alpha)} b_{r,s,\mathrm{i,j}} \ge \varsigma\right\} \subseteq \left\{(r,s) \in \mathbb{N}^2 : \sum_{\mathbb{t}_{\mathrm{i,j}}\notin \mathrm{O}_{\curlyvee}} b_{nk} \ge \varsigma - \curlyvee\right\}.$$

Since the sequence $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to $\alpha \in X$, the right hand side set belongs to $\mathcal{I}_2$. Therefore, (2) is true.

On the other hand, suppose that (2) is true and let $B \in \mathcal{B}_\alpha$. Given that $\mathcal{E} \in \mathfrak{J}(\mathfrak{H})$, there is a $\mathfrak{M} > 0$ such that $\mathcal{E}(\beta,\alpha) > \mathfrak{M}$ for any $\beta \notin B$. Then, we have

$$\sum_{\mathbb{t}_{i,j}\notin B} b_{r,s,i,j} \leq \frac{1+\mathfrak{M}}{\mathfrak{M}} \sum_{\mathbb{t}_{i,j}\notin B} \frac{\mathcal{E}(\mathbb{t}_{i,j},\alpha)}{1+\mathcal{E}(\mathbb{t}_{i,j},\alpha)} b_{r,s,i,j} \leq \frac{1+\mathfrak{M}}{\mathfrak{M}} \sum_{i,j} \frac{\mathcal{E}(\mathbb{t}_{i,j},\alpha)}{1+\mathcal{E}(\mathbb{t}_{i,j},\alpha)} b_{r,s,i,j}.$$

Therefore, for any $\varsigma > 0$ choose $\curlyvee = \frac{\sigma\cdot\mathfrak{M}}{1+\mathfrak{M}}$ and then

$$\left\{(r,s) \in \mathbb{N}^2 : \sum_{\mathbb{t}_{i,j}\notin B} b_{r,s,i,j} \geq \varsigma\right\} \subseteq \left\{(r,s) \in \mathbb{N}^2 : \sum_{i,j} \frac{\mathcal{E}(\mathbb{t}_{i,j},\alpha)}{1+\mathcal{E}(\mathbb{t}_{i,j},\alpha)} b_{r,s,i,j} \geq \curlyvee\right\}.$$

Now (2) gives that the right hand side set belongs to $\mathcal{I}_2$ and hence the sequence $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to $\alpha \in \mathfrak{H}$.

The characterisation of $\mathfrak{A}^{\mathcal{I}_2}$-statistical convergence of real sequences easily leads to the aforementioned Theorem.

Corollary 2.2. Assume that $\mathfrak{A} = (b_{r,s,i,j}) \in \lfloor NN - R - S \rfloor$ and $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is a real sequence in $\mathbb{R}^2$ with the usual topology. Then $\{\mathbb{t}_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is $\mathfrak{A}^{\mathcal{I}_2}$-statistically convergent to a real number $\mathbb{J}$ iff

$$\mathcal{I}_2 - \lim_{r,s} \sum_{i,j} \frac{|\mathbb{t}_{i,j} - \mathbb{J}|}{1+|\mathbb{t}_{i,j} - \mathbb{J}|} b_{r,s,i,j} = 0.$$

III. CONCLUSION

A new approach to strong convergence, termed $\mathfrak{A}_{\mathcal{T}}^{\mathcal{I}_2}$-strong convergence, is introduced here by employing a class of pre-metrics with properties analogous to those based on arbitrary Hausdorff topological spaces. This method addresses the limitations of linearity. In the context of this new form of strong convergence, we explore its relationships with $\mathfrak{A}^{\mathcal{I}_2}$-statistical convergence. Finally, we provide a partial characterization of $\mathfrak{A}^{\mathcal{I}_2}$-statistical convergence.

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