Uluslararası İleri Doğa Bilimleri ve Mühendislik Araştırmaları Dergisi
Sayı 8, S. 200-210, 7, 2024

Araştırma Makalesi

International Journal of Advanced Natural Sciences and Engineering Researches
Volume 8, pp. 200-210, 7, 2024

Research Article

https://as-proceeding.com/index.php/ijanser
ISSN: 2980-0811

Some Results on Generalized Statistical Convergence of Double Sequences via Ideals in Probabilistic Generalized Metric Spaces

Ömer Kişi [1*], Mehmet Gürdal [2] and Elif Hevesker [3]

[1]*Department of Mathematics, Bartın University, Bartın, Turkey*
[2] *Department of Mathematics, Suleyman Demirel University, Isparta, Turkey*
[3] *Department of Mathematics, Bartın University, Bartın, Turkey*

[*]*(okisi@bartin.edu.tr) Email of the corresponding author*

(Received: 22 August 2024, Accepted: 28 August 2024)

(5th International Conference on Engineering and Applied Natural Sciences ICEANS 2024, August 25-26, 2024)

ATIF/REFERENCE: Kişi, Ö., Gürdal, M. & Hevesker, E. (2024). Some Results on Generalized Statistical Convergence of Double Sequences via Ideals in Probabilistic Generalized Metric Spaces. *International Journal of Advanced Natural Sciences and Engineering Researches*, 8(7), 200-210.

***Abstract* –** For double sequences in probabilistic generalized metric spaces (PGMS), we establish the notions of $\mathfrak{I}_2$-statistical convergence and $\mathfrak{I}_2$-statistical Cauchyness in this work and investigate their fundamental properties, including their interrelationships.

Keywords – Probabilistic Generalized Metric Space, $\mathfrak{I}_2$-Statistical Convergence, $\mathfrak{I}_2$-Statistical Cauchyness.

I. INTRODUCTION

In 1951, Fast [9] and later, in 1959, Schoenberg [22], independently developed and examined the theory of statistical convergence of real sequences. They based their work on the concept of asymptotic density of subsets within the set of natural numbers $\mathbb{N}$. Convergence for double sequences was first proposed by Pringsheim in [21]. Mursaleen and Edely [18] went on to expand this idea to statistical convergence.

The concept of $\mathfrak{I}$-convergence was first proposed by Kostyrko et al. [16] as a generalization of statistical convergence. Gürdal introduced the concept of $\mathfrak{I}$-Cauchy sequences of real numbers, examining their connections with $\mathfrak{I}$-convergence for sequences of real numbers (see also [10]). In the same thesis, he defined the concepts of $\mathfrak{I}$-Cauchy sequences and $\mathfrak{I}^*$-Cauchy sequences in a metric space (X, d) (see also [10]). Additionally, he explored the relationships between $\mathfrak{I}$-convergence and $\mathfrak{I}$-Cauchy sequences. Das et al. [4] later adapted this idea to double sequences within a metric space, highlighting several properties of this form of convergence. Building on this, Das et al. [5] further advanced the concept, evolving it into $\mathfrak{I}$-statistical convergence. Additional research in this area can be found in studies conducted by [6,7,8,11,14].

Definition 1.1. ([16]) Let $\mathfrak{Z} \neq \emptyset$. A non-void class $\mathfrak{I} \subset 2^{\mathfrak{Z}}$ is stated to be an ideal, if

(a) $\mathcal{E}, \mathcal{F} \in \mathfrak{I} \Rightarrow \mathcal{E} \cup \mathcal{F} \in \mathfrak{I}$

(b) $\mathcal{E} \in \mathfrak{I}, \mathcal{F} \subset \mathcal{E} \Rightarrow \mathcal{E} \in \mathfrak{I}$.

Definition 1.2. ([16]) An ideal $\mathfrak{I}$ will be termed non-trivial in $\mathfrak{Z}$ if $\mathfrak{I} \neq \{\emptyset\}$ and $\mathfrak{Z} \notin \mathfrak{I}$.

Definition 1.3. ([16]) An ideal $\mathfrak{I}$ in $\mathfrak{Z}$ is stated to be admissible if it is non-trivial and for each $y \in \mathfrak{Z}, \{y\} \in \mathfrak{I}$.

Definition 1.4. ([16]) For each ideal $\mathfrak{I}$ in $\mathfrak{Z}$, we can associate a filter called the associated filter $\mathcal{F}(\mathfrak{I}) = \{M \subset \mathfrak{Z}: \mathfrak{Z} \setminus N = M \text{ for some } N \in \mathfrak{I}\}$.

Definition 1.5. ([5]) A real number sequence (ω_u) is stated to be $\mathfrak{I}$-statistically convergent to ω if for each ϱ and $0 < \varsigma < 1$

$$\left\{s: \frac{1}{s} |\{u \leq s: |\omega_u - \omega| \geq \varrho\}| \geq \varsigma\right\} \in \mathfrak{I}.$$

In this scenario, we will put $\mathfrak{I} - st - \lim_{u \to \infty} \omega_u = \omega$

In 1942, Menger [17] explored the concept of "probabilistic metric spaces (PMS)" by using a distribution function F_{ab} to define the distance between two points a and b, rather than relying on a real number. The function $F_{ab}(t)$, where $t > 0$, denotes the probability that the distance between a and b is less than t. Building on Menger's pioneering work, numerous scholars, such as Schwiezer and Sklar [23, 24] and Tardiff [26], have advanced the theory of probabilistic metric spaces. For additional information, consult the comprehensive book on probabilistic metric spaces [25].

The theory of PGMS builds upon the concept of G-metric spaces. Recent studies have made substantial contributions to understanding generalized statistical convergence [15], asymptotically lacunary statistically equivalent sequences [12], and the convergence of double sequences in G-metric spaces. For additional information on G-metric spaces, refer to [3, 13, 19, 20].

In 2014, Zhou et al. [27] introduced and explored the theory of PGMS, an extension of PMS. Given the well-established uses of PMS, it is expected that PGMS will also prove to be highly applicable in the future. For recent research on PGMS, refer to [1, 28]. We will now revisit the definition of PGMS as presented in [27].

Now, let's recall the definition of PGMS from [27].

Definition 1.6. ([27]). Let Θ be a nonempty set, $\mathfrak{Y}$ be a function from $\Theta \times \Theta \times \Theta$ into $\mathcal{D}^+$and δ be a continuous t-norm such that for each $\alpha, \beta, \gamma \in \Theta$, we have

(1) $\mathfrak{Y}_{(\alpha,\beta,\gamma)}(q) = 1,$ for all $\alpha, \beta, \gamma \in \Theta$ and $q > 0$ if and only if $\alpha = \beta = \gamma$;

(2) $\mathfrak{Y}_{(\alpha,\alpha,\beta)}(q) \geq \mathfrak{Y}_{(\alpha,\beta,\gamma)}(q)$ for each $\alpha, \beta, \gamma(\neq \beta)$, and $q > 0$;

(3) $\mathfrak{Y}_{(\alpha,\beta,\gamma)}(q) = \mathfrak{Y}_{(\beta,\alpha,\gamma)}(q) = \mathfrak{Y}_{(\gamma,\alpha,\beta)}(q) = \cdots$ (symmetry in $\alpha, \beta, \gamma \in \Theta$);

(4) $\mathfrak{Y}_{(\alpha,\beta,\gamma)}(u + v) \geq \delta\big(\mathfrak{Y}_{(\alpha,w,w)}(u), \mathfrak{Y}_{(w,\beta,\gamma)}(v)\big)$ for each $\alpha, \beta, \gamma, w \in \Theta$ and $u, v \geq 0$.

Then, $(\Theta, \mathfrak{Y}, \delta)$ is referred to as a Menger PGMS (in short a PGMS).

Let $\alpha_0 \in \Theta$. Then, for each ϱ and $0 < \varsigma < 1$ the (ϱ, ς)-neighbourhood of a_0 is given by $\mathfrak{i}_{a_0}(\rho, \sigma)$ and is described as

$$\mathfrak{a}_{a_0}(\varrho, \varsigma) = \left\{\beta \in \Theta : \mathfrak{G}_{(\alpha_0, \beta, \beta)}(\varrho) > 1 - \varsigma, \mathfrak{G}_{(\beta, \alpha_0, \alpha_0)}(\varrho) > 1 - \varsigma\right\}.$$

Given that various notions of sequence convergence are crucial for investigating the topological characteristics of topological spaces, and considering that the topological properties of Menger PGMS have yet to be extensively explored, this article introduces and investigates the theory of $\mathfrak{I}_2$-statistical convergence sequences within probabilistic generalized metric spaces.

Definition 1.7. ([27]) Suppose that (ω_u) is a sequence in a PGMS $(\Theta, \mathfrak{Y}, \delta)$ and $\omega \in \Theta$. Then, Θ is stated to be

(1) converges to ω if for each ϱ and $0 < \varsigma < 1$, there is a non-zero non-negative integer $M_{\varrho,\varsigma}$ such that

$$\omega_u \in \mathfrak{i}_\omega(\varrho, \varsigma) \text{ whenever } u \geq M_{\varrho,\varsigma}.$$

(2) Cauchy if for each ϱ and $0 < \varsigma < 1$, there is an $M_{\varrho,\varsigma} \in \mathbb{N}$ such that

$$\mathfrak{Y}_{(\omega_u, \omega_i, \omega_l)}(\varrho) > 1 - \varsigma \text{ whenever } u, i, l \geq M_{\varrho,\varsigma}.$$

Definition 1.8. ([1]) Suppose that (ω_u) is a sequence in a PGMS $(\Theta, \mathfrak{Y}, \delta)$. Then (ω_u) is stated to be statistically

(1) converges to ω if for each $q > 0$,

$$d(\{u : \omega_u \notin \mathfrak{i}_\omega(q)\}) = 0.$$

(2) Cauchy if for each $q > 0$, there is $l_q \in \mathbb{N}$ such that

$$d\left(\left\{u : \omega_u \notin \mathfrak{i}_{w_{l_q}}(q)\right\}\right) = 0.$$

II. $\mathfrak{I}_2$-STATISTICAL CONVERGENCE IN PGMS

In this section, we institute and examine the theory of $\mathfrak{I}_2$-statistically Cauchy sequence.

Definition 2.2. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS. A sequence (ω_{uv}) in Θ is considered $\mathfrak{I}_2$-statistically convergent to $\omega \in \Theta$ if, for each ϱ and $0 < \varsigma < 1$ and $\kappa > 0$,

$$\left\{(\alpha, \beta) \in \mathbb{N}^2 : \frac{1}{\alpha\beta} |\{u \leq \alpha, v \leq \beta : \omega_{uv} \notin \mathfrak{i}_\omega(\varrho, \varsigma)\}| \geq \kappa\right\} \in \mathfrak{I}_2,$$

to put it differently,

$$d^{\mathfrak{I}_2}(\{(u, v) : \omega_{uv} \notin \mathfrak{i}_\omega(\varrho, \varsigma)\}) = 0.$$

In this context, we express $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v \to \infty} \omega_{uv} = \omega$.

Remark 2.3. Every statistically convergent sequence is also $\mathfrak{I}_2$-statistically convergent in a PGMS $(\Theta, \mathfrak{Y}, \tau)$.

Proposition 2.4. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS. If (ω_{uv}) is a sequence in Θ such that $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\omega_{uv} = \omega_1$ and $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\omega_{uv} = \omega_2$, then $\omega_1 = \omega_2$.

Theorem 2.5. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS and $\mathfrak{I}_2$ be an nontrivial admissible ideal of $\mathbb{N}^2$. Let $(\omega_{uv}), (\gamma_{uv})$ and (η_{uv}) be sequences in Θ and $\omega, \gamma, \zeta \in \Theta$. If $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\omega_{uv} = \omega$, $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\gamma_{uv} = \gamma$, and $\mathfrak{I}_2 - st^{\mathfrak{G}} - \lim_{u,v\to\infty}\eta_{uv} = \eta$, then the sequence $\left(\mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\varrho)\right)$ is $\mathfrak{I}_2$-statistically convergent to $\mathfrak{Y}_{\omega,\gamma,\zeta}(\varrho)$ for each ϱ.

Proof. Let ϱ be given. Choose $\varsigma > 0$ such that $\varrho - 2\varsigma > 0$. Then we get

$$\begin{aligned}
&\mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\varrho)\\
&\geq \mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\varrho - \varsigma)\\
&\geq \delta\left(\mathfrak{Y}_{(\omega_{uv,},\omega,\omega)}\left(\frac{\varsigma}{3}\right), \mathfrak{Y}_{(\omega,\gamma_{uv},\eta_{uv})}\left(\frac{3\varrho - 4\varsigma}{3}\right)\right)\\
&\geq \delta\left(\mathfrak{Y}_{(\omega_{uv,},\omega,\omega)}\left(\frac{\varsigma}{3}\right), \tau\left(\mathfrak{Y}_{(\gamma_{uv},\gamma,\gamma)}\left(\frac{\varsigma}{3}\right), \mathfrak{Y}_{(\gamma,\omega,\eta_{uv})}\left(\varrho - \frac{5\varsigma}{3}\right)\right)\right)\\
&\geq \delta\left(\mathfrak{Y}_{(\omega_{uv,},\omega,\omega)}\left(\frac{\varsigma}{3}\right), \tau\left(\mathfrak{Y}_{(\gamma_{uv},\gamma,\gamma)}\left(\frac{\varsigma}{3}\right), \tau\left(\mathfrak{Y}_{(\eta_{uv},\eta,\eta)}\left(\frac{\varsigma}{3}\right), \mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho - 2\varsigma)\right)\right)\right).
\end{aligned}$$

Additionally, the following is held:

$$\begin{aligned}
&\mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho)\\
&\geq \mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho - \sigma)\\
&\geq \delta\left(\mathfrak{Y}_{(\omega,\omega_{uv},\omega_{uv})}\left(\frac{\varsigma}{3}\right), \mathfrak{Y}_{(\omega_{uv},\gamma,\eta)}\left(\frac{3\varrho - 4\varsigma}{3}\right)\right)\\
&\geq \delta\left(\mathfrak{Y}_{(\omega,\omega_{uv},\omega_{uv})}\left(\frac{\varsigma}{3}\right), \tau\left(\mathfrak{Y}_{(\gamma,\gamma_{uv},\gamma_{uv})}\left(\frac{\varsigma}{3}\right), \mathfrak{Y}_{(\gamma_{uv},\omega_{uv,},\eta)}\left(\varrho - \frac{5\varsigma}{3}\right)\right)\right)\\
&\geq \delta\left(\mathfrak{Y}_{(\omega,\omega_{uv},\omega_{uv})}\left(\frac{\varsigma}{3}\right), \tau\left(\mathfrak{Y}_{(\gamma,\gamma_{uv},\gamma_{uv})}\left(\frac{\varsigma}{3}\right), \tau\left(\mathfrak{Y}_{(\eta,\gamma_{uv},\gamma_{uv})}\left(\frac{\varsigma}{3}\right), \mathfrak{Y}_{(\omega_{uv,},\gamma_{uv},\eta_{uv})}(\varrho - 2\varsigma)\right)\right)\right).
\end{aligned}$$

Since δ is continuous, it follows from [2, Theorem 2] that it is statistically continuous. Therefore, there exist sets $\mathcal{A}$ and $\mathcal{B}$ of non-zero non-negative integers with $\mathfrak{I}_2$-density 1 such that $\mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\varrho - 2\varsigma) \geq \mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho)$ for all $(u, v) \in \mathcal{A}$, and $\mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\rho) \geq \mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho - 2\varsigma)$ for all $(u, v) \in \mathcal{B}$. Set $C = \mathcal{A} \cap \mathcal{B}$. Then $d^{I_2}(C) = 1$. In addition, $\mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\varrho - 2\varsigma) \geq \mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho)$ and $\mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\varrho) \geq \mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho - 2\varsigma)$ for all $(u, v) \in C$. Since $\mathfrak{Y}$ is left-continuous, it follows that $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\mathfrak{Y}_{\omega_{uv},\gamma_{uv},\eta_{uv}}(\varrho) = \mathfrak{Y}_{(\omega,\gamma,\eta)}(\varrho)$ for each ϱ.

Theorem 2.6. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS and (ω_{uv}) be a sequence in Θ. The condition $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\omega_{uv} = \omega$ is equivalent to the existence of a subset $\mathfrak{X} = \{(k_u, l_v): k_u < k_{u+1}, l_v < l_{v+1}\}$ of $\mathbb{N}^2$ such that $d^{\mathfrak{I}_2}(\mathfrak{X}) = 1$ and $\mathfrak{Y} - \lim_{u,v\to\infty}\omega_{k_u l_v} = \omega$.

Proof. Let there is a subset $\mathfrak{X} = \{(k_u, l_v): k_u < k_{u+1}, l_v < l_{v+1}\}$ of $\mathbb{N}^2$ such that $d^{\mathfrak{I}_2}(\mathfrak{X}) = 1$ and $\mathfrak{Y} - \lim_{u,v\to\infty}\omega_{k_u l_v} = \omega$. Let ϱ and $0 < \varsigma < 1$ be given. Then, there is a non-zero non-negative integer $n_{\varrho,\varsigma}$ such that $\omega_{k_u l_v} \in \mathfrak{i}_\omega(\varrho, \varsigma)$ whenever $u, v \geq n_{\varrho,\varsigma}$. Thus

$$\{(u,v): \omega_{uv} \in \mathfrak{i}_\omega(\varrho,\varsigma)\} \supset \left\{(k_u, l_v): u, v \geq n_{\varrho,\varsigma}\right\}.$$

Since the latter set has $\mathfrak{I}_2$-density 1, $d^{\mathfrak{I}_2}(\{(u,v): \omega_{uv} \in \mathfrak{i}_\omega(\varrho,\varsigma)\}) = 1$. Hence, $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\omega_{uv} = \omega$.

Conversely, let $\mathfrak{I}_2 - st^{\mathfrak{Y}} - \lim_{u,v\to\infty}\omega_{uv} = \omega$. Then, for each ϱ and $0 < \varsigma < 1$,

$$d^{\mathfrak{I}_2}(\{(u,v): \omega_{uv} \in \mathfrak{i}_\omega(\varrho,\varsigma)\}) = 1.$$

Set

$$\mathcal{E}(\varrho,\varsigma) = \{(u,v): \omega_{uv} \in \mathfrak{i}_\omega(\varrho,\varsigma)\}$$

for each ϱ and $0 < \varsigma < 1$. Clearly, $d^{\mathfrak{I}_2}\big(\mathcal{E}(\varrho,\varsigma)\big) = 1$. Now for $\varrho_{\alpha\beta} = \frac{1}{\alpha\beta}$ and $\varsigma_{\alpha\beta} = \frac{1}{\alpha\beta}$ with $\alpha, \beta \geq 2$, we have $\mathfrak{i}_\omega\left(\frac{1}{2},\frac{1}{2}\right) \supset \mathfrak{i}_\omega\left(\frac{1}{3},\frac{1}{3}\right) \supset \cdots \supset \mathfrak{i}_\omega\left(\frac{1}{\alpha\beta},\frac{1}{\alpha\beta}\right) \supset \mathfrak{i}_\omega\left(\frac{1}{\alpha\beta+1},\frac{1}{\alpha\beta+1}\right) \supset \cdots$

Consequently,

$$\mathcal{E}(1/2,1/2) \supset \mathcal{E}(1/3,1/3) \supset \cdots \supset \mathcal{E}(1/\alpha\beta, 1/\alpha\beta) \supset \mathcal{E}(1/(\alpha\beta+1),1/(\alpha\beta+1)) \supset \cdots$$

Note that $d^{\mathfrak{I}_2}(\mathcal{E}(1/\alpha\beta, 1/\alpha\beta)) = 1$ for each $\alpha, \beta (> 1) \in \mathbb{N}$. Set $t_1 = 1$. Since $d^{\mathfrak{I}_2}(\mathcal{E}(1/2,1/2)) = 1$, there is $t_2 \in \mathcal{E}(1/2,1/2)$ and $t_2 > t_1$ such that for each $\alpha, \beta \geq t_2$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta: (u,v) \in \mathcal{E}(1/2,1/2)\}|}{\alpha\beta} > 1 - 1/2.$$

Since $d^{\mathfrak{I}_2}(\mathcal{E}(1/3,1/3)) = 1$, there is $t_3 \in \mathcal{E}(1/3,1/3)$ with $t_3 > t_2$ such that for each $\alpha, \beta \geq t_3$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta: (u,v) \in \mathbb{E}(1/3,1/3)\}|}{\alpha\beta} > 1 - 1/3.$$

Again, since $d^{\mathfrak{I}_2}(\mathcal{E}(1/4,1/4)) = 1$, there is $t_4 \in v(1/4,1/4)$ with $t_4 > t_3$ such that $\forall\alpha, \beta \geq t_4$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta: (u,v) \in \mathcal{E}(1/4,1/4)\}|}{\alpha\beta} > 1 - 1/4.$$

Continue in this manner, we will get a strictly increasing sequence of nonzero non-negative integers (t_m) such that $t_m \in \mathcal{E}(1/m, 1/m)$ and for each $\alpha, \beta \geq t_m$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{E}(1/m, 1/m)\}|}{\alpha\beta} > 1 - 1/m.$$

We now construct a set $\mathcal{A}$ as follows:

$$\mathcal{A} = \{(u,v) : u, v \in [t_1, t_2]\} \bigcup \left\{ \bigcup_{m \in \mathbb{N}} \{(u,v) : u, v \in [t_m, t_{m+1}] \cap \mathcal{E}(1/m, 1/m)\} \right\}.$$

Then, for each $r \in \mathbb{N}$ with $t_m \leq r < t_{m+1}$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{A}\}|}{\alpha\beta} \geq \frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{E}(1/m, 1/m)\}|}{\alpha\beta} \geq 1 - \frac{1}{m}.$$

Thus, $d^{\mathfrak{I}_2}(\mathcal{A}) = 1$. Let ϱ and $0 < \varsigma < 1$. We choose a large $q \in \mathbb{N}$ such that

$$\frac{1}{q} < \varrho \text{ and } \frac{1}{q} < \varsigma.$$

Let $u, v \geq t_q$, and $r \in \mathcal{A}$. Then, there is $j \in \mathbb{N}$ such that $t_j \leq u, v < t_{j+1}$ and $j > q$. Clearly, $(u,v) \in \mathcal{A}\left(\frac{1}{j}, \frac{1}{j}\right)$. Thus,

$$\omega_{uv} \in \mathfrak{i}_\omega\left(\frac{1}{j}, \frac{1}{j}\right) \subset \mathfrak{i}_\omega\left(\frac{1}{q}, \frac{1}{q}\right) \subset \mathfrak{i}_\omega(\varrho, \varsigma).$$

Therefore $\omega_{uv} \in \mathfrak{i}_\omega(\varrho, \varsigma)$ for each $(u,v) \in \mathcal{A}$ with $u, v \geq t_q$. Write $\mathcal{A} = \{(k_u, l_v) : k_u < k_{u+1}, l_v < l_{v+1}\}$. Hence $\mathfrak{Y} - \lim_{u,v\to\infty} \omega_{k_u l_v} = \omega$.

Corollary 2.7. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS and (ω_{uv}) be a sequence in Θ. Then, $\omega_{uv} \overset{\mathfrak{I}_2 - st^{\mathfrak{Y}}}{\rightarrow} \omega$ if and only if there exists a sequence (ζ_{uv}) such that $\omega_{uv} = \zeta_{uv}$ for almost all u,v $(\mathfrak{I}_2)$ and $\zeta_{uv} \overset{\mathfrak{Y}}{\rightarrow} \omega$.

We now provide a necessary condition for a sequence to be $\mathfrak{I}_2$-statistically convergent.

Theorem 2.8. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS and (ω_{uv}) be a sequence in Θ. If (ω_{uv}) is $\mathfrak{I}_2$-statistically convergent to Θ, then for each ϱ and $0 < \varsigma < 1$, there is a non-zero non-negative integer $i_0 = i_0(\varrho, \varsigma)$ such that

$$\mathfrak{I}_2 - \lim_{\alpha,\beta\to\infty} \frac{1}{\alpha\beta}\left|\left\{u \leq \alpha, v \leq \beta : \mathfrak{Y}_{(\omega_{uv}, \omega_{i_0}, \omega_{i_0})}(\varrho) > 1 - \varsigma, \mathfrak{Y}_{(\omega_{i_0}, \omega_{uv}, \omega_{uv})}(\varrho) > 1 - \varsigma\right\}\right| = 1.$$

Proof. Let ϱ as well as $0 < \varsigma < 1$ be given. Since δ is continuous, there is $\varsigma_0 \in (0,1)$ such that

$$\delta(1 - \varsigma_0, 1 - \varsigma_0) > 1 - \varsigma.$$

Again, since $\omega_{uv} \overset{\mathfrak{I}_2 - st^{\mathfrak{Y}}}{\rightarrow} \omega$, we have $d^{\mathfrak{I}_2}\left(\left\{(u,v) : \omega_{uv} \in \mathfrak{i}_\omega\left(\frac{\varrho}{2}, \varsigma_0\right)\right\}\right) = 1$. Set

$$\mathcal{A}=\left\{(u,v):\omega_{uv}\in \mathfrak{i}_{\omega}\left(\frac{\varrho}{2},\varsigma_0\right)\right\}.$$

Clearly, $d^{\mathfrak{I}_2}(\mathcal{A})=1$. Let n_0 be an arbitrary but fixed element of $\mathbb{A}$. Then $\omega_{i_0}\in \mathfrak{i}_{\omega}\left(\frac{\varrho}{2},\varsigma_0\right)$. Then for each $(u,v)\in\mathcal{A}$, we have

$$\begin{aligned}&\mathfrak{Y}_{(\omega_{uv},\omega_{i_0},\omega_{i_0})}(\varrho)\\ \geq\ &\delta\left(\mathfrak{Y}_{(\omega_{uv},\omega,\omega)}\left(\frac{\varrho}{2}\right),\mathfrak{Y}_{(\omega,\omega_{i_0},\omega_{i_0})}\left(\frac{\varrho}{2}\right)\right)\\ >\ &\delta(1-\varsigma_0,1-\varsigma_0)>1-\varsigma.\end{aligned}$$

Also,

$$\begin{aligned}&\mathfrak{Y}_{(\omega_{i_0},\omega_{uv},\omega_{uv})}(\varrho)\\ \geq&\delta\left(\mathfrak{Y}_{(\omega_{i_0},\omega,\omega)}\left(\frac{\varrho}{2}\right),\mathfrak{Y}_{(\omega,\omega_{uv},\omega_{uv})}\left(\frac{\varrho}{2}\right)\right)\\ >&\delta(1-\varsigma_0,1-\varsigma_0)>1-\varsigma.\end{aligned}$$

Therefore,

$$\mathcal{A}\subset\left\{(u,v):\mathfrak{Y}_{(\omega_{uv},\omega_{i_0},\omega_{i_0})}(\varrho)>1-\varsigma,\mathfrak{Y}_{(\omega_{i_0},\omega_{uv},\omega_{uv})}(\varrho)>1-\varsigma\right\}.$$

Thus,

$$\mathfrak{I}_2-\lim_{\alpha,\beta\to\infty}\frac{1}{\alpha\beta}\left|\left\{u\leq\alpha,v\leq\beta:\mathfrak{Y}_{(\omega_{uv},\omega_{i_0},\omega_{i_0})}(\varrho)>1-\varsigma,\mathfrak{Y}_{(\omega_{i_0},\omega_{uv},\omega_{uv})}(\varrho)>1-\varsigma\right\}\right|=1.$$

Let ϱ as well as $0<\varsigma<1$ be given. Since δ is continuous, there is $\varsigma_0\in(0,1)$ such that

$$\delta(1-\varsigma_0,1-\varsigma_0)>1-\varsigma.$$

Again, since $\omega_{uv}\overset{\mathfrak{I}_2-st^{\mathfrak{Y}}}{\rightarrow}\omega$, we have $d^{\mathfrak{I}_2}\left(\left\{(u,v):\omega_{uv}\in \mathfrak{i}_{\omega}\left(\frac{\varrho}{2},\varsigma_0\right)\right\}\right)=1$. Set

$$\mathcal{A}=\left\{(u,v):\omega_{uv}\in \mathfrak{i}_{\omega}\left(\frac{\varrho}{2},\varsigma_0\right)\right\}$$

Clearly, $d^{\mathfrak{I}_2}(\mathcal{A})=1$. Let n_0 be an arbitrary but fixed element of $\mathcal{A}$. Then $x_{i_0}\in \mathfrak{i}_{x}\left(\frac{\rho}{2},\sigma_0\right)$. Then for each $k\in\mathcal{A}$, we have

$$\begin{aligned}&\mathfrak{Y}_{(\omega_{uv},\omega_{i_0},\omega_{i_0})}(\varrho)\\ \geq\quad&\delta\left(\mathfrak{Y}_{(\omega_{uv},\omega,\omega)}\left(\frac{\varrho}{2}\right),\mathfrak{Y}_{(\omega,\omega_{i_0},\omega_{i_0})}\left(\frac{\varrho}{2}\right)\right)\\ >\quad&\delta(1-\varsigma_0,1-\varsigma_0)>1-\varsigma.\end{aligned}$$

Also,

$$\begin{aligned}&\mathfrak{Y}_{(\omega_{i_0},\omega_{uv},\omega_{uv})}(\varrho)\\ \geq&\delta\left(\mathfrak{Y}_{(\omega_{i_0},\omega,\omega)}\left(\frac{\varrho}{2}\right),\mathfrak{Y}_{(\omega,\omega_{uv},\omega_{uv})}\left(\frac{\varrho}{2}\right)\right)\\ >&\delta(1-\varsigma_0,1-\varsigma_0)>1-\varsigma.\end{aligned}$$

Therefore,

$$\mathcal{A}\subset\left\{(u,v):\mathfrak{Y}_{(\omega_{uv},\omega_{i_0},\omega_{i_0})}(\varrho)>1-\varsigma,\mathfrak{Y}_{(\omega_{i_0},\omega_{uv},\omega_{uv})}(\varrho)>1-\varsigma\right\}.$$

Thus, $\mathfrak{I}_2-\lim_{\alpha,\beta\to\infty}\frac{1}{\alpha\beta}\left|\left\{u\leq\alpha,v\leq\beta:\mathfrak{Y}_{(\omega_{uv},\omega_{i_0},\omega_{i_0})}(\varrho)>1-\varsigma,\mathfrak{Y}_{(\omega_{i_0},\omega_{uv},\omega_{uv})}(\varrho)>1-\varsigma\right\}\right|=1.$

III. $\mathfrak{I}_2$-STATISTICAL CAUCHYNESS IN PGMS

Definition 3.1. Consider $(\Theta,\mathfrak{Y},\delta)$ as a PGMS and let (ω_{uv}) be a sequence in Θ. Then, (ω_{uv}) is called $\mathfrak{I}_2$-statistically Cauchy if, for each ϱ and $0<\varsigma<1$, there exist $n_{\varrho,\varsigma},m_{\varrho,\varsigma}\in\mathbb{N}$ such that

$$d^{\mathfrak{I}_2}\left(\left\{(u,v):\omega_{uv}\notin\text{ı}_{\omega_{u_{n_{\varrho,\varsigma}}v_{m_{\varrho,\varsigma}}}}(\varrho,\varsigma)\right\}\right)=0.$$

Remark 3.2. If (ω_{uv}) is a statistically Cauchy sequence in a PGMS Θ, then it is also $\mathfrak{I}_2$-statistically Cauchy sequence in Θ.

Theorem 3.3. Consider $(\Theta,\mathfrak{Y},\delta)$ as a PGMS and let (ω_{uv}) be a sequence in Θ. Then (ω_{uv}) is $\mathfrak{I}_2$-statistically Cauchy in Θ if and only if there is a subset $\mathfrak{P}=\{(k_u,l_v):k_u<k_{u+1},l_v<l_{v+1}\}$ of $\mathbb{N}^2$ such that $d^{\mathfrak{I}_2}(\mathfrak{P})=1$ as well as $(\omega)_{\mathbb{P}}$ is a Cauchy sequence.

Proof. Consider a subset $\mathfrak{P}=\{(k_u,l_v):k_u<k_{u+1},l_v<l_{v+1}\}$ of $\mathbb{N}^2$ such that $d^{\mathfrak{I}_2}(\mathfrak{P})=1$ and $(\omega)_{\mathfrak{P}}$ is a Cauchy sequence. Let ϱ and $0<\varsigma<1$ be given. Then, there exist non-zero non-negative $n_{\varrho,\varsigma},m_{\varrho,\varsigma}$ such that $\omega_{k_ul_v}\in\text{ı}_{\omega_{u_{n_{\varrho,\varsigma}}v_{m_{\varrho,\varsigma}}}}(\varrho,\varsigma)$ whenever $\geq n_{\varrho,\varsigma}$, $v\geq m_{\varrho,\varsigma}$. Thus

$$\left\{(u,v):\omega_{uv}\in\text{ı}_{\omega_{u_{n_{\varrho,\varsigma}}v_{m_{\varrho,\varsigma}}}}(\varrho,\varsigma)\right\}\supset\left\{(k_u,l_v):u\geq n_{\varrho,\varsigma},v\geq m_{\varrho,\varsigma}\right\}.$$

Given that the set has $\mathfrak{I}_2$-density 1, it follows that $d^{\mathfrak{I}_2}\left(\left\{(u,v):\omega_{uv}\in\text{ı}_{\omega_{u_{n_{\varrho,\varsigma}}v_{m_{\varrho,\varsigma}}}}(\varrho,\varsigma)\right\}\right)=1$. Therefore, (ω_{uv}) is $\mathfrak{I}_2$-statistically Cauchy.

Conversely, (ω_{uv}) is $\mathfrak{I}_2$-statistically Cauchy. Then for each ρ as well as $0<\sigma<1$

$$d^{\mathfrak{I}_2}\left(\left\{(u,v):\omega_{uv}\in\text{ı}_{\omega_{u_{n_{\varrho,\varsigma}}v_{m_{\varrho,\varsigma}}}}(\varrho,\varsigma)\right\}\right)=1.$$

Set $\mathcal{E}(\rho,\sigma)=\left\{(u,v):\omega_{uv}\in\text{ı}_{\omega_{u_{n_{\varrho,\varsigma}}v_{m_{\varrho,\varsigma}}}}(\varrho,\varsigma)\right\}$ for each ϱ and $0<\varsigma<1$. Clearly, $d^{\mathfrak{I}_2}(\mathcal{E}(\rho,\sigma))=1$.

Now for $\varrho_{\alpha\beta}=\frac{1}{\alpha\beta}$ and $\varsigma_{\alpha\beta}=\frac{1}{\alpha\beta}$ with $\alpha,\beta\geq2$, we have $\text{ı}_\omega\left(\frac{1}{2},\frac{1}{2}\right)\supset\text{ı}_\omega\left(\frac{1}{3},\frac{1}{3}\right)\supset\cdots\supset\text{ı}_\omega\left(\frac{1}{\alpha\beta},\frac{1}{\alpha\beta}\right)\supset\text{ı}_\omega\left(\frac{1}{\alpha\beta+1},\frac{1}{\alpha\beta+1}\right)\supset\cdots$

Consequently,

$$\mathcal{E}(1/2,1/2) \supset \mathcal{E}(1/3,1/3) \supset \cdots \supset \mathcal{E}(1/\alpha\beta, 1/\alpha\beta) \supset \mathcal{E}(1/(\alpha\beta+1),1/(\alpha\beta+1)) \supset \cdots$$

Note that $d^{\mathfrak{I}_2}(\mathcal{E}(1/\alpha\beta, 1/\alpha\beta)) = 1$ for each $\alpha, \beta(>1) \in \mathbb{N}$. Set $t_1 = 1$. Since $d^{\mathfrak{I}_2}(\mathcal{E}(1/2,1/2)) = 1$, there is $t_2 \in \mathcal{E}(1/2,1/2)$ and $t_2 > t_1$ such that for each $\alpha, \beta \geq t_2$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{E}(1/2,1/2)\}|}{\alpha\beta} > 1 - 1/2.$$

Since $d^{\mathfrak{I}_2}(\mathcal{E}(1/3,1/3)) = 1$, there is $t_3 \in \mathcal{E}(1/3,1/3)$ with $t_3 > t_2$ such that for each $\alpha, \beta \geq t_3$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{E}(1/3,1/3)\}|}{\alpha\beta} > 1 - 1/3.$$

Again, since $d^{\mathfrak{I}_2}(\mathcal{E}(1/4,1/4)) = 1$, there is $t_4 \in \mathcal{E}(1/4,1/4)$ with $t_4 > t_3$ such that $\forall \alpha, \beta \geq t_4$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{E}(1/4,1/4)\}|}{\alpha\beta} > 1 - 1/4.$$

Continue in this manner, we will get a strictly increasing sequence of nonzero non-negative integers (t_m) such that $t_m \in \mathcal{E}(1/m, 1/m)$ and for each $\alpha, \beta \geq t_m$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{E}(1/m, 1/m)\}|}{\alpha\beta} > 1 - 1/m.$$

We now construct a set $\mathcal{A}$ as follows:

$$\mathcal{A} = \{(u,v) : u,v \in [t_1, t_2]\} \bigcup \left\{ \bigcup_{m \in \mathbb{N}} \{(u,v) : u,v \in [t_m, t_{m+1}] \cap \mathbb{E}(1/m, 1/m)\} \right\}.$$

Then, for each $r \in \mathbb{N}$ with $t_m \leq r < t_{m+1}$, we have

$$\frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{A}\}|}{\alpha\beta} \geq \frac{|\{u \leq \alpha, v \leq \beta : (u,v) \in \mathcal{E}(1/m, 1/m)\}|}{\alpha\beta} \geq 1 - \frac{1}{m}.$$

Thus, $d^{\mathfrak{I}_2}(\mathcal{A}) = 1$. Let ϱ and $0 < \varsigma < 1$. We choose a large $q \in \mathbb{N}$ such that

$$\frac{1}{q} < \varrho \text{ and } \frac{1}{q} < \varsigma.$$

Let $u, v \geq t_q$, and $r \in \mathcal{A}$. Then, there is $j \in \mathbb{N}$ such that $t_j \leq u, v < t_{j+1}$ and $j > q$. Clearly, $(u,v) \in \mathcal{A}\left(\frac{1}{j}, \frac{1}{j}\right)$. Thus,

$$\omega_{uv} \in \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}} v_{m_{\varrho,\varsigma}}}}\left(\frac{1}{j}, \frac{1}{j}\right) \subset \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}} v_{m_{\varrho,\varsigma}}}}\left(\frac{1}{q}, \frac{1}{q}\right) \subset \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}} v_{m_{\varrho,\varsigma}}}}(\varrho, \varsigma).$$

Therefore $\omega_{uv} \in \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}}v_{m_{\varrho,\varsigma}}}}(\varrho,\varsigma)$ for each $(u,v) \in \mathcal{A}$ with $u,v \geq t_q$. Write $\mathcal{A} = \{(k_u, l_v): k_u < k_{u+1}, l_v < l_{v+1}\}$. Hence, $(\omega)_{\mathfrak{P}}$ is a Cauchy sequence.

Corollary 3.4. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS, and let (ω_{uv}) be a sequence in Θ. Then, (ω_{uv}) is a Cauchy sequence in Θ implies and implied by there is a sequence (q_{uv}) such that $\omega_{uv} = q_{uv}$ for almost all u,v $(\mathfrak{I}_2)$ and (q_{uv}) is also a Cauchy sequence in Θ.

Theorem 3.5. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS. If (ω_{uv}) is an $\mathfrak{I}_2$-statistically convergent sequence in Θ, then (ω_{uv}) is $\mathfrak{I}_2$-statistically Cauchy in Θ.

Proof. The proof follows directly from Theorem 2.8.

Corollary 3.6. Let $(\Theta, \mathfrak{Y}, \delta)$ be a PGMS and (ω_{uv}) be an $\mathfrak{I}_2$-statistically convergent sequence in Θ. Then, there is a subset $\mathfrak{P} = \{(k_u, l_v): k_u < k_{u+1}, l_v < l_{v+1}\}$ of $\mathbb{N}^2$ such that $d^{\mathfrak{I}_2}(\mathfrak{P}) = 1$ as well as $(\omega)_{\mathfrak{P}}$ is a Cauchy sequence.

IV. CONCLUSION

We have presented and examined the fundamental properties and links between the notions of $\mathfrak{I}_2$-statistical convergence and $\mathfrak{I}_2$-statistical Cauchyness for double sequences within probabilistic generalized metric spaces (PGMS) in this work. Our findings reveal that these concepts extend traditional convergence notions into the probabilistic framework of PGMS, establishing a clear connection between $\mathfrak{I}_2$-statistical convergence and $\mathfrak{I}_2$-statistical Cauchyness. This work enriches the theoretical understanding of sequence behavior in PGMS and sets the stage for further exploration of these ideas in more complex mathematical contexts.

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