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Some Results For Probabilistic Generalized Metric Spaces

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Abstract – In this research, we examine some fundamental characteristics of λ-statistical convergence in probabilistic generalized metric spaces ($\mathcal{PGMS}$). Additionally, we define and explore the concept of λ-statistical Cauchyness and investigate their interrelationships.

Keywords – Probabilistic Generalized Metric Space, λ-Statistical Convergence, λ-Statistical Cauchyness.

I. INTRODUCTION

For many years, one of the most important and active areas of research in pure mathematics has been the study of sequence convergence and summability theory. Moreover, it has made significant contributions to a wide range of fields, including computer science, mathematical modeling, functional analysis, topology, measure theory, and applied mathematics. In recent years, the idea of statistical convergence of sequences has been applied widely in mathematics. Fast [2] conducted a first exploration of statistical convergence. Since then, a number of mathematicians have investigated the statistical convergence and convergence features and applied these ideas to many disciplines.

Mursaleen [11] further extended the concept of natural density of subsets of $\mathbb{N}$ to the concept of λ-density. Using λ-density, he further extended the concept of statistical convergence of real sequences to the concept of λ-statistical convergence. If $\lambda = \{\lambda_u\}_{u\in\mathbb{N}}$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_1 = 1, \lambda_{u+1} \leq \lambda_u + 1, u \in \mathbb{N}$, then any subset $\mathfrak{C}$ of $\mathbb{N}$ is said to have λ-density $d^\lambda(\mathfrak{C})$ if

$$d^\lambda(\mathfrak{C}) = \lim_{u\to\infty} \frac{|\{k \in I_u : k \in \mathfrak{C}\}|}{\lambda_u},$$

where $I_u = [u - \lambda_u + 1, u]$. It is clear that if $\mathfrak{U}, \mathfrak{B} \subset \mathbb{N}$ and $d^\lambda(\mathfrak{U}) = 0$, $d^\lambda(\mathfrak{B}) = 0$ then $d^\lambda(\mathfrak{U}^c) = 1 = d^\lambda(\mathfrak{B}^c)$, $d^\lambda(\mathfrak{U} \cup \mathfrak{B}) = 0$. Also if $\mathfrak{U}, \mathfrak{B} \subset \mathbb{N}$, $\mathfrak{U} \subset \mathfrak{B}$ and $d^\lambda(\mathfrak{B}) = 0$, then $d^\lambda(\mathfrak{U}) = 0$. In this study, λ represents such a sequence.

If a sequence $\varpi = \{\varpi_u\}_{u\in\mathbb{N}}$ is said to satisfy the condition $\mathcal{P}$ for "λ-almost all u," or more succinctly, "λ-$a.a.u.$", if it satisfies the property $\mathcal{P}$ for all u with the exception of a set of λ-density zero. When $d^\lambda(\mathfrak{C}(\varepsilon)) = 0$ for each $\varepsilon > 0$, where $\mathfrak{C}(\varepsilon) = \{u \in \mathbb{N} : |\varpi_u - \varsigma| \geq \varepsilon\}$, then a series of real numbers, $\varpi =$

$\{\varpi_u\}_{u\in\mathbb{N}}$, is said to be λ-statistically convergent or S_λ -convergent to $\varsigma \in \mathbb{R}$. The concepts of λ-density and λ-statistical convergence correspond with the ideas of natural density and statistical convergence, respectively, if $\lambda_u = u, \forall u \in \mathbb{N}$.

Instead of utilizing a real integer to determine the distance between two locations, $\mathcal{a}$ and $\mathcal{b}$, Menger [10] investigated the idea of "probabilistic metric spaces ($\mathcal{PMS}$)" in 1942. When $t > 0$, the function F_{ab} indicates the likelihood that a and b's distance is smaller than t. The likelihood that the distance between a and b is shorter than t is indicated by the function $F_{ab}(t)$, where $t > 0$. Many researchers, including Tardiff [17] and Schwiezer and Sklar [14, 15], have expanded the theory of probabilistic metric spaces by building on Menger's groundbreaking work. Refer to the extensive work on probabilistic metric spaces [16] for further details.

G-metric spaces are the foundation of $\mathcal{PGMS}$ theory. Understanding asymptotically lacunary statistically equivalent sequences [6], extended statistical convergence [9], and the convergence of double sequences in G-metric spaces have all benefited from recent research. Further reading on G-metric spaces may be found in [3, 7, 12, 13].

Zhou et al. [18] presented and investigated the hypothesis of $\mathcal{PGMS}$, which is an expansion of $\mathcal{PMS}$, in 2014. It is anticipated that $\mathcal{PGMS}$ will show to be extremely useful in the future, given the known uses of $\mathcal{PMS}$. Regarding current studies on $\mathcal{PGMS}$, see [1, 19]. Let's go back to the definition of PGMS that was given in [18].

Let's now review the definition of $\mathcal{PGMS}$ found in [18].

Definition 1.1. ([18]). Let Ξ be a nonempty set, $\mathfrak{Y}$ be a function from $\Xi \times \Xi \times \Xi$ into $\mathcal{D}^+$and δ be a continuous t-norm such that for each $\alpha, \beta, \gamma \in \Xi$, we have

(1) $\mathfrak{Y}_{(\alpha,\beta,\gamma)}(q) = 1,$ for all $\alpha, \beta, \gamma \in \Xi$ and $q > 0$ iff $\alpha = \beta = \gamma$;

(2) $\mathfrak{Y}_{(\alpha,\alpha,\beta)}(q) \geq \mathfrak{Y}_{(\alpha,\beta,\gamma)}(q)$ for each $\alpha, \beta, \gamma (\neq \beta)$, and $q > 0$;

(3) $\mathfrak{Y}_{(\alpha,\beta,\gamma)}(q) = \mathfrak{Y}_{(\beta,\alpha,\gamma)}(q) = \mathfrak{Y}_{(\gamma,\alpha,\beta)}(q) = \cdots$ (symmetry in $\alpha, \beta, \gamma \in \Xi$);

(4) $\mathfrak{Y}_{(\alpha,\beta,\gamma)}(u + v) \geq \delta\big(\mathfrak{Y}_{(\alpha,w,w)}(u), \mathfrak{Y}_{(w,\beta,\gamma)}(v)\big)$ for each $\alpha, \beta, \gamma, w \in \Xi$ and $u, v \geq 0$.

Then, $(\Xi, \mathfrak{Y}, \delta)$ is referred to as a Menger $\mathcal{PGMS}$ (in short a $\mathcal{PGMS}$).

Let $\alpha_0 \in \Xi$. Then, for each ϱ and $0 < \varsigma < 1$ the (ϱ, ς)-neighbourhood of a_0 is given by $\text{ɩ}_{a_0}(\rho, \sigma)$ and is described as

$$\mathfrak{a}_{a_0}(\varrho, \varsigma) = \big\{\beta \in \Xi : \mathfrak{G}_{(\alpha_0,\beta,\beta)}(\varrho) > 1 - \varsigma, \mathfrak{G}_{(\beta,\alpha_0,\alpha_0)}(\varrho) > 1 - \varsigma\big\}.$$

This paper presents and explores the theory of λ-statistical convergence sequences within probabilistic generalized metric spaces, considering that different concepts of sequence convergence are essential for examining the topological features of topological spaces and that the topological properties of Menger $\mathcal{PGMS}$ have not been thoroughly studied.

Definition 1.2. ([18]) Suppose that (ω_u) is a sequence in a $\mathcal{PGMS}$ $(\Xi, \mathfrak{Y}, \delta)$ and $\varpi \in \Xi$. Then, Ξ is stated to be

(i) converges to ϖ if for each ϱ and $0 < \varsigma < 1$, there is a non-zero non-negative integer $\mathfrak{M}_{\varrho,\varsigma}$ such that

$$\omega_u \in \mathfrak{i}_{\varpi}(\varrho, \varsigma) \text{ whenever } u \geq \mathfrak{M}_{\varrho,\varsigma}.$$

(ii) Cauchy if for each ϱ and $0 < \varsigma < 1$, there is an $\mathfrak{M}_{\varrho,\varsigma} \in \mathbb{N}$ such that

$$\mathfrak{Y}_{(\omega_u,\omega_i,\omega_l)}(\varrho) > 1 - \varsigma \text{ whenever } u, i, l \geq \mathfrak{M}_{\varrho,\varsigma}.$$

Definition 1.3. ([1]) Suppose that (ω_u) is a sequence in a $\mathcal{PGMS}$ $(\Xi, \mathfrak{Y}, \delta)$. Then (ω_u) is stated to be statistically

(i) converges to ϖ if for each $q > 0$,

$$d(\{u: \omega_u \notin \mathfrak{i}_{\varpi}(q)\}) = 0.$$

(ii) Cauchy if for each $q > 0$, there is $l_q \in \mathbb{N}$ such that

$$d\left(\left\{u: \omega_u \notin \mathfrak{i}_{w_{l_q}}(q)\right\}\right) = 0.$$

II. THE MAIN RESULTS

We define and examine λ-statistical convergence as well as λ-statistical Cauchy sequence theory in this chapter.

Definition 2.1. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$. A sequence (ω_u) in Ξ is considered λ-statistically convergent to $\varpi \in \Xi$ if, for each ϱ and $0 < \varsigma < 1$ and $\kappa > 0$,

$$d^{\lambda}(\{u \leq \alpha : \omega_u \notin \mathfrak{i}_{\varpi}(\varrho, \varsigma)\}) = 0.$$

In this context, we express $\lambda - st^{\mathfrak{Y}} - \lim_{u\to\infty}\omega_u = \varpi$.

Proposition 2.2. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$. If (ω_u) is a sequence in Ξ such that $\lambda - st^{\mathfrak{Y}} - \lim_{u\to\infty}\omega_u = \varpi_1$ and $\lambda - st^{\mathfrak{Y}} - \lim_{u\to\infty}\omega_u = \varpi_2$, then $\varpi_1 = \varpi_2$.

Theorem 2.3. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$. Let $(\omega_u), (\gamma_u)$ and (η_u) be sequences in Ξ and $\varpi, \gamma, \zeta \in \Xi$. If $\lambda - st^{\mathfrak{Y}} - \lim_{u\to\infty}\omega_u = \varpi$, $\lambda - st^{\mathfrak{Y}} - \lim_{u\to\infty}\gamma_u = \gamma$, and $\lambda - st^{\mathfrak{G}} - \lim_{u\to\infty}\eta_u = \eta$, then the sequence $\left(\mathfrak{Y}_{\omega_{u},\gamma_u,\eta_u}(\varrho)\right)$ is λ-statistically convergent to $\mathfrak{Y}_{\varpi,\gamma,\zeta}(\varrho)$ for each ϱ.

Proof. Let ϱ be given. Choose $\varsigma > 0$ such that $\varrho - 2\varsigma > 0$. Then we get

$$
\begin{aligned}
&\mathfrak{Y}_{\omega_u,\gamma_u,\eta_u}(\varrho)\\
\geq &\mathfrak{Y}_{\omega_u,\gamma_u,\eta_u}(\varrho-\varsigma)\\
\geq &\delta\left(\mathfrak{Y}_{(\omega_u,\omega,\omega)}\left(\frac{\varsigma}{3}\right),\mathfrak{Y}_{(\varpi,\gamma_u,\eta_u)}\left(\frac{3\varrho-4\varsigma}{3}\right)\right)\\
\geq &\delta\left(\mathfrak{Y}_{(\omega_u,\varpi,\varpi)}\left(\frac{\varsigma}{3}\right),\tau\left(\mathfrak{Y}_{(\gamma_u,\gamma,\gamma)}\left(\frac{\varsigma}{3}\right),\mathfrak{Y}_{(\gamma,\varpi,\eta_u)}\left(\varrho-\frac{5\varsigma}{3}\right)\right)\right)\\
\geq &\delta\left(\mathfrak{Y}_{(\omega_u,\varpi,\varpi)}\left(\frac{\varsigma}{3}\right),\tau\left(\mathfrak{Y}_{(\gamma_u,\gamma,\gamma)}\left(\frac{\varsigma}{3}\right),\tau\left(\mathfrak{Y}_{(\eta_u,\eta,\eta)}\left(\frac{\varsigma}{3}\right),\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho-2\varsigma)\right)\right)\right).
\end{aligned}
$$

Additionally, the following is held:

$$
\begin{aligned}
&\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho)\\
\geq &\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho-\sigma)\\
\geq &\delta\left(\mathfrak{Y}_{(\varpi,\omega_u,\omega_u)}\left(\frac{\varsigma}{3}\right),\mathfrak{Y}_{(\omega_u,\gamma,\eta)}\left(\frac{3\varrho-4\varsigma}{3}\right)\right)\\
\geq &\delta\left(\mathfrak{Y}_{(\varpi,\omega_u,\omega_u)}\left(\frac{\varsigma}{3}\right),\tau\left(\mathfrak{Y}_{(\gamma,\gamma_u,\gamma_u)}\left(\frac{\varsigma}{3}\right),\mathfrak{Y}_{(\gamma_u,\omega_u,\eta)}\left(\varrho-\frac{5\varsigma}{3}\right)\right)\right)\\
\geq &\delta\left(\mathfrak{Y}_{(\varpi,\omega_u,\omega_u)}\left(\frac{\varsigma}{3}\right),\tau\left(\mathfrak{Y}_{(\gamma,\gamma_u,\gamma_u)}\left(\frac{\varsigma}{3}\right),\tau\left(\mathfrak{Y}_{(\eta,\gamma_u,\gamma_u)}\left(\frac{\varsigma}{3}\right),\mathfrak{Y}_{(\omega_u,\gamma_u,\eta_u)}(\varrho-2\varsigma)\right)\right)\right).
\end{aligned}
$$

Since δ is continuous, it follows from Theorem 2 in [2] that it is statistically continuous. Therefore, there exist sets $\mathcal{A}$ and $\mathcal{B}$ of non-zero non-negative integers with λ-density 1 such that $\mathfrak{Y}_{\omega_u,\gamma_u,\eta_u}(\varrho-2\varsigma)\geq\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho)$ for all $u\in\mathcal{A}$, and $\mathfrak{Y}_{\omega_u,\gamma_u,\eta_u}(\rho)\geq\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho-2\varsigma)$ for all $u\in\mathcal{B}$. Set $C=\mathcal{A}\cap\mathcal{B}$. Then $d^{\lambda}(C)=1$. In addition, $\mathfrak{Y}_{\omega_u,\gamma_u,\eta_u}(\varrho-2\varsigma)\geq\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho)$ and $\mathfrak{Y}_{\omega_u,\gamma_u,\eta_u}(\varrho)\geq\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho-2\varsigma)$ for all $u\in C$. Since $\mathfrak{Y}$ is left-continuous, it follows that $\lambda-st^{\mathfrak{Y}}-\lim_{u\to\infty}\mathfrak{Y}_{\omega_u,\gamma_u,\eta_u}(\varrho)=\mathfrak{Y}_{(\varpi,\gamma,\eta)}(\varrho)$ for each ϱ.

Theorem 2.4. Let $(\Xi,\mathfrak{Y},\delta)$ be a $\mathcal{PGMS}$ and (ω_u) be a sequence in Ξ. The condition $\lambda-st^{\mathfrak{Y}}-\lim_{u\to\infty}\omega_u=\varpi$ is equivalent to the existence of a subset $\mathfrak{X}=\{k_u\colon k_u<k_{u+1}\}$ of $\mathbb{N}$ such that $d^{\lambda}(\mathfrak{X})=1$ and $\mathfrak{Y}-\lim_{u\to\infty}\omega_{k_u}=\varpi$.

Proof. Let there is a subset $\mathfrak{X}=\{k_u\colon k_u<k_{u+1}\}$ of $\mathbb{N}$ such that $d^{\lambda}(\mathfrak{X})=1$ and $\mathfrak{Y}-\lim_{u\to\infty}\omega_{k_u}=\varpi$. Let ϱ and $0<\varsigma<1$ be given. Then, there is a non-zero non-negative integer $n_{\varrho,\varsigma}$ such that $\omega_{k_u}\in\mathfrak{i}_{\varpi}(\varrho,\varsigma)$ whenever $u\geq n_{\varrho,\varsigma}$. Thus

$$\{u\colon\omega_u\in\mathfrak{i}_{\varpi}(\varrho,\varsigma)\}\supset\{k_u\colon u\geq n_{\varrho,\varsigma}\}.$$

Since the latter set has d^{λ}-density 1, $d^{\lambda}(\{u\colon\omega_u\in\mathfrak{i}_{\varpi}(\varrho,\varsigma)\})=1$. Hence, $d^{\lambda}-st^{\mathfrak{Y}}-\lim_{u\to\infty}\omega_u=\varpi$.

Conversely, let $\lambda-st^{\mathfrak{Y}}-\lim_{u\to\infty}\omega_u=\varpi$. Then, for each ϱ and $0<\varsigma<1$,

$$d^{\lambda}(\{u\colon\omega_u\in\mathfrak{i}_{\varpi}(\varrho,\varsigma)\})=1.$$

Set

$$\mathcal{E}(\varrho,\varsigma)=\{u\colon \omega_u\in \mathrm{i}_{\varpi}(\varrho,\varsigma)\}$$

for each ϱ and $0<\varsigma<1$. Clearly, $d^{\mathfrak{I}_2}\big(\mathcal{E}(\varrho,\varsigma)\big)=1$. Now for $\varrho_\alpha=\frac{1}{\alpha}$ and $\varsigma_\alpha=\frac{1}{\alpha}$ with $\alpha\geq 2$, we have $\mathrm{i}_\omega\left(\frac{1}{2},\frac{1}{2}\right)\supset \mathrm{i}_\omega\left(\frac{1}{3},\frac{1}{3}\right)\supset\cdots\supset \mathrm{i}_\omega\left(\frac{1}{\alpha},\frac{1}{\alpha}\right)\supset \mathrm{i}_\omega\left(\frac{1}{\alpha+1},\frac{1}{\alpha+1}\right)\supset\cdots$

Consequently,

$$\mathcal{E}(1/2,1/2)\supset\mathcal{E}(1/3,1/3)\supset\cdots\supset\mathcal{E}(1/\alpha,1/\alpha)\supset\mathcal{E}(1/(\alpha+1),1/(\alpha+1))\supset\cdots$$

Note that $d^{\lambda}(\mathcal{E}(1/\alpha,1/\alpha))=1$ for each $\alpha(>1)\in\mathbb{N}$. Set $t_1=1$. Since $d^{\lambda}(\mathcal{E}(1/2,1/2))=1$, there is $t_2\in\mathcal{E}(1/2,1/2)$ and $t_2>t_1$ such that for each $\alpha\geq t_2$, we have

$$\frac{|\{u\leq\alpha\colon \mathrm{u}\in\mathcal{E}(1/2,1/2)\}|}{\alpha}>1-1/2.$$

Since $d^{\lambda}(\mathcal{E}(1/3,1/3))=1$, there is $t_3\in\mathcal{E}(1/3,1/3)$ with $t_3>t_2$ such that for each $\alpha\geq t_3$, we have

$$\frac{|\{u\leq\alpha\colon \mathrm{u}\in\mathcal{E}(1/3,1/3)\}|}{\alpha}>1-1/3.$$

Again, since $d^{\lambda}(\mathcal{E}(1/4,1/4))=1$, there is $t_4\in v(1/4,1/4)$ with $t_4>t_3$ such that $\forall\alpha\geq t_4$, we have

$$\frac{|\{u\leq\alpha\colon \mathrm{u}\in\mathcal{E}(1/4,1/4)\}|}{\alpha}>1-1/4.$$

Continue in this manner, we will get a strictly increasing sequence of nonzero non-negative integers (t_m) such that $t_m\in\mathcal{E}(1/m,1/m)$ and for each $\alpha\geq t_m$, we have

$$\frac{|\{u\leq\alpha\colon \mathrm{u}\in\mathcal{E}(1/m,1/m)\}|}{\alpha}>1-1/m.$$

We now construct a set $\mathcal{A}$ as follows:

$$\mathcal{A}=\{u\colon u\in[t_1,t_2]\}\bigcup\left\{\bigcup_{m\in\mathbb{N}}\{u\colon u\in[t_m,t_{m+1}]\cap\mathcal{E}(1/m,1/m)\}\right\}.$$

Then, for each $r\in\mathbb{N}$ with $t_m\leq r<t_{m+1}$, we have

$$\frac{|\{u\leq\alpha\colon \mathrm{u}\in\mathcal{A}\}|}{\alpha}\geq\frac{|\{u\leq\alpha\colon \mathrm{u}\in\mathcal{E}(1/m,1/m)\}|}{\alpha}\geq 1-\frac{1}{m}.$$

Thus, $d^{\lambda}(\mathcal{A})=1$. Let ϱ and $0<\varsigma<1$. We choose a large $q\in\mathbb{N}$ such that

$$\frac{1}{q}<\varrho \text{ and } \frac{1}{q}<\varsigma.$$

Let $u \geq t_q$, and $r \in \mathcal{A}$. Then, there is $j \in \mathbb{N}$ such that $t_j \leq u < t_{j+1}$ and $j > q$. Clearly, $u \in \mathcal{A}\left(\frac{1}{j}, \frac{1}{j}\right)$. Thus,

$$\omega_u \in \mathfrak{i}_\varpi\left(\frac{1}{j}, \frac{1}{j}\right) \subset \mathfrak{i}_\varpi\left(\frac{1}{q}, \frac{1}{q}\right) \subset \mathfrak{i}_\varpi(\varrho, \varsigma).$$

Therefore $\omega_u \in \mathfrak{i}_\varpi(\varrho, \varsigma)$ for each $u \in \mathcal{A}$ with $u \geq t_q$. Write $\mathcal{A} = \{k_u : k_u < k_{u+1}\}$. Hence $\mathfrak{Y} - \lim_{u\to\infty} \omega_{k_u} = \varpi$.

Corollary 2.5. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$ and (ω_u) be a sequence in Ξ. Then, $\omega_u \overset{\lambda-st^{\mathfrak{Y}}}{\to} \varpi$ iff there exists a sequence (ζ_u) such that $\omega_u = \zeta_u$ for almost all u (λ) and $\zeta_u \overset{\mathfrak{Y}}{\to} \varpi$.

We now provide a necessary condition for a sequence to be λ-statistically convergent.

Theorem 2.6. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$ and (ω_u) be a sequence in Ξ. If (ω_u) is λ-statistically convergent to ϖ, then for each ϱ and $0 < \varsigma < 1$, there is a non-zero non-negative integer $i_0 = i_0(\varrho, \varsigma)$ such that

$$\lambda - \lim_{\alpha\to\infty} \frac{1}{\alpha}\left|\left\{u \leq \alpha : \mathfrak{Y}_{(\omega_u, \omega_{i_0}, \omega_{i_0})}(\varrho) > 1 - \varsigma, \mathfrak{Y}_{(\omega_{i_0}, \omega_u, \omega_u)}(\varrho) > 1 - \varsigma\right\}\right| = 1.$$

Proof. Let ϱ as well as $0 < \varsigma < 1$ be given. Since δ is continuous, there is $\varsigma_0 \in (0,1)$ such that

$$\delta(1 - \varsigma_0, 1 - \varsigma_0) > 1 - \varsigma.$$

Again, since $\omega_u \overset{\lambda-st^{\mathfrak{Y}}}{\to} \varpi$, we have $d^\lambda\left(\left\{u : \omega_u \in \mathfrak{i}_\varpi\left(\frac{\varrho}{2}, \varsigma_0\right)\right\}\right) = 1$. Set

$$\mathcal{A} = \left\{u : \omega_u \in \mathfrak{i}_\varpi\left(\frac{\varrho}{2}, \varsigma_0\right)\right\}.$$

Clearly, $d^\lambda(\mathcal{A}) = 1$. Let n_0 be an arbitrary but fixed element of $\mathcal{A}$. Then $\omega_{i_0} \in \mathfrak{i}_\varpi\left(\frac{\varrho}{2}, \varsigma_0\right)$. Then for each $u \in \mathcal{A}$, we have

$$\begin{aligned} &\mathfrak{Y}_{(\omega_u, \omega_{i_0}, \omega_{i_0})}(\varrho) \\ \geq\ &\delta\left(\mathfrak{Y}_{(\omega_u, \varpi, \varpi)}\left(\frac{\varrho}{2}\right), \mathfrak{Y}_{(\varpi, \omega_{i_0}, \omega_{i_0})}\left(\frac{\varrho}{2}\right)\right) \\ >\ &\delta(1 - \varsigma_0, 1 - \varsigma_0) > 1 - \varsigma. \end{aligned}$$

Also,

$$\begin{aligned} &\mathfrak{Y}_{(\omega_{i_0}, \omega_u, \omega_u)}(\varrho) \\ \geq &\delta\left(\mathfrak{Y}_{(\omega_{i_0}, \varpi, \varpi)}\left(\frac{\varrho}{2}\right), \mathfrak{Y}_{(\varpi, \omega_u, \omega_u)}\left(\frac{\varrho}{2}\right)\right) \\ > &\delta(1 - \varsigma_0, 1 - \varsigma_0) > 1 - \varsigma. \end{aligned}$$

Therefore,

$$\mathcal{A} \subset \left\{u: \mathfrak{Y}_{(\omega_u,\omega_{i_0},\omega_{i_0})}(\varrho) > 1-\varsigma, \mathfrak{Y}_{(\omega_{i_0},\omega_u,\omega_u)}(\varrho) > 1-\varsigma\right\}.$$

Thus,

$$\lambda - \lim_{\alpha\to\infty} \frac{1}{\alpha}\left|\left\{u \leq \alpha: \mathfrak{Y}_{(\omega_u,\omega_{i_0},\omega_{i_0})}(\varrho) > 1-\varsigma, \mathfrak{Y}_{(\omega_{i_0},\omega_u,\omega_u)}(\varrho) > 1-\varsigma\right\}\right| = 1.$$

Let ϱ as well as $0<\varsigma<1$ be given. Since δ is continuous, there is $\varsigma_0 \in (0,1)$ such that

$$\delta(1-\varsigma_0, 1-\varsigma_0) > 1-\varsigma.$$

Again, since $\omega_u \overset{\lambda-st^{\mathfrak{Y}}}{\rightarrow} \varpi$, we have $d^\lambda\left(\left\{u: \omega_u \in \mathfrak{i}_\varpi\left(\frac{\varrho}{2},\varsigma_0\right)\right\}\right) = 1$. Set

$$\mathcal{A} = \left\{u: \omega_u \in \mathfrak{i}_\varpi\left(\frac{\varrho}{2},\varsigma_0\right)\right\}$$

Clearly, $d^\lambda(\mathcal{A}) = 1$. Let n_0 be an arbitrary but fixed element of $\mathcal{A}$. Then $x_{i_0} \in \mathfrak{i}_x\left(\frac{\rho}{2},\sigma_0\right)$. Then for each $k \in \mathcal{A}$, we have

$$\begin{aligned} & \mathfrak{Y}_{(\omega_u,\omega_{i_0},\omega_{i_0})}(\varrho) \\ \geq\ & \delta\left(\mathfrak{Y}_{(\omega_u,\varpi,\varpi)}\left(\frac{\varrho}{2}\right), \mathfrak{Y}_{(\varpi,\omega_{i_0},\omega_{i_0})}\left(\frac{\varrho}{2}\right)\right) \\ >\ & \delta(1-\varsigma_0, 1-\varsigma_0) > 1-\varsigma. \end{aligned}$$

Also,

$$\begin{aligned} & \mathfrak{Y}_{(\omega_{i_0},\omega_u,\omega_u)}(\varrho) \\ \geq & \delta\left(\mathfrak{Y}_{(\omega_{i_0},\varpi,\varpi)}\left(\frac{\varrho}{2}\right), \mathfrak{Y}_{(\varpi,\omega_u,\omega_u)}\left(\frac{\varrho}{2}\right)\right) \\ > & \delta(1-\varsigma_0, 1-\varsigma_0) > 1-\varsigma. \end{aligned}$$

Therefore,

$$\mathcal{A} \subset \left\{u: \mathfrak{Y}_{(\omega_u,\omega_{i_0},\omega_{i_0})}(\varrho) > 1-\varsigma, \mathfrak{Y}_{(\omega_{i_0},\omega_u,\omega_u)}(\varrho) > 1-\varsigma\right\}.$$

Thus, $\lambda - \lim_{\alpha\to\infty} \frac{1}{\alpha}\left|\left\{u \leq \alpha: \mathfrak{Y}_{(\omega_u,\omega_{i_0},\omega_{i_0})}(\varrho) > 1-\varsigma, \mathfrak{Y}_{(\omega_{i_0},\omega_u,\omega_u)}(\varrho) > 1-\varsigma\right\}\right| = 1.$

We can now give the concept of λ -statistical Cauchyness in $\mathcal{PGMS}$ and related results.

Definition 2.7. Consider $(\Xi, \mathfrak{Y}, \delta)$ as a $\mathcal{PGMS}$ and let (ω_u) be a sequence in Ξ. Then, (ω_u) is called λ-statistically Cauchy if, for each ϱ and $0<\varsigma<1$, there exist $n_{\varrho,\varsigma} \in \mathbb{N}$ such that

$$d^\lambda\left(\left\{u: \omega_u \notin \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho,\varsigma)\right\}\right) = 0.$$

Theorem 2.8. Consider $(\Xi, \mathfrak{Y}, \delta)$ as a $\mathcal{PGMS}$ and let (ω_u) be a sequence in Θ. Then (ω_u) is λ-statistically Cauchy in Ξ iff there is a subset $\mathfrak{P} = \{k_u: k_u < k_{u+1}\}$ of $\mathbb{N}$ such that $d^\lambda(\mathfrak{P}) = 1$ as well as $(\omega)_{\mathfrak{P}}$ is a Cauchy sequence.

Proof. Consider a subset $\mathfrak{P} = \{k_u: k_u < k_{u+1}\}$ of $\mathbb{N}$ such that $d^\lambda(\mathfrak{P}) = 1$ and $(\omega)_{\mathfrak{P}}$ is a Cauchy sequence. Let ϱ and $0 < \varsigma < 1$ be given. Then, there exist non-zero non-negative $n_{\varrho,\varsigma}$ such that $\omega_{k_u} \in \text{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho, \varsigma)$ whenever $u \geq n_{\varrho,\varsigma}$. Thus

$$\left\{u: \omega_u \in \text{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho, \varsigma)\right\} \supset \{k_u: u \geq n_{\varrho,\varsigma}\}.$$

Given that the set has λ-density 1, it follows that $d^\lambda\left(\left\{u: \omega_u \in \text{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho, \varsigma)\right\}\right) = 1$. Therefore, (ω_u) is λ-statistically Cauchy.

Conversely, (ω_u) is λ-statistically Cauchy. Then for each ρ as well as $0 < \sigma < 1$

$$d^\lambda\left(\left\{u: \omega_u \in \text{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho, \varsigma)\right\}\right) = 1.$$

Set $\mathcal{E}(\rho, \sigma) = \left\{u: \omega_u \in \text{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho, \varsigma)\right\}$ for each ϱ and $0 < \varsigma < 1$. Clearly, $d^\lambda(\mathcal{E}(\rho, \sigma)) = 1$.

Now for $\varrho_\alpha = \frac{1}{\alpha}$ and $\varsigma_\alpha = \frac{1}{\alpha}$ with $\alpha \geq 2$, we have $\text{i}_\omega\left(\frac{1}{2}, \frac{1}{2}\right) \supset \text{i}_\omega\left(\frac{1}{3}, \frac{1}{3}\right) \supset \cdots \supset \text{i}_\omega\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right) \supset \text{i}_\omega\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right) \supset \cdots$

Consequently,

$$\mathcal{E}(1/2,1/2) \supset \mathcal{E}(1/3,1/3) \supset \cdots \supset \mathcal{E}(1/\alpha, 1/\alpha) \supset \mathcal{E}(1/(\alpha+1),1/(\alpha+1)) \supset \cdots$$

Note that $d^\lambda(\mathcal{E}(1/\alpha, 1/\alpha)) = 1$ for each $\alpha(> 1) \in \mathbb{N}$. Set $t_1 = 1$. Since $d^\lambda(\mathcal{E}(1/2,1/2)) = 1$, there is $t_2 \in \mathcal{E}(1/2,1/2)$ and $t_2 > t_1$ such that for each $\alpha \geq t_2$, we have

$$\frac{|\{u \leq \alpha: \text{u} \in \mathcal{E}(1/2,1/2)\}|}{\alpha} > 1 - 1/2.$$

Since $d^\lambda(\mathcal{E}(1/3,1/3)) = 1$, there is $t_3 \in \mathcal{E}(1/3,1/3)$ with $t_3 > t_2$ such that for each $\alpha \geq t_3$, we have

$$\frac{|\{u \leq \alpha: \text{u} \in \mathcal{E}(1/3,1/3)\}|}{\alpha} > 1 - 1/3.$$

Again, since $d^\lambda(\mathcal{E}(1/4,1/4)) = 1$, there is $t_4 \in \mathcal{E}(1/4,1/4)$ with $t_4 > t_3$ such that $\forall\alpha \geq t_4$, we have

$$\frac{|\{u \leq \alpha: \text{u} \in \mathcal{E}(1/4,1/4)\}|}{\alpha} > 1 - 1/4.$$

Continue in this manner, we will get a strictly increasing sequence of nonzero non-negative integers (t_m) such that $t_m \in \mathcal{E}(1/m, 1/m)$ and for each $\alpha \geq t_m$, we have

$$\frac{|\{u \leq \alpha : \mathrm{u} \in \mathcal{E}(1/m, 1/m)\}|}{\alpha} > 1 - 1/m.$$

We now construct a set $\mathcal{A}$ as follows:

$$\mathcal{A} = \{u : u \in [t_1, t_2]\} \bigcup \left\{ \bigcup_{m \in \mathbb{N}} \{u : u \in [t_m, t_{m+1}] \cap \mathcal{E}(1/m, 1/m)\} \right\}.$$

Then, for each $r \in \mathbb{N}$ with $t_m \leq r < t_{m+1}$, we have

$$\frac{|\{u \leq \alpha : \mathrm{u} \in \mathcal{A}\}|}{\alpha} \geq \frac{|\{u \leq \alpha : \mathrm{u} \in \mathcal{E}(1/m, 1/m)\}|}{\alpha} \geq 1 - \frac{1}{m}.$$

Thus, $d^{\lambda}(\mathcal{A}) = 1$. Let ϱ and $0 < \varsigma < 1$. We choose a large $q \in \mathbb{N}$ such that

$$\frac{1}{q} < \varrho \text{ and } \frac{1}{q} < \varsigma.$$

Let $u \geq t_q$, and $r \in \mathcal{A}$. Then, there is $j \in \mathbb{N}$ such that $t_j \leq u < t_{j+1}$ and $j > q$. Clearly, $\mathrm{u} \in \mathcal{A}\left(\frac{1}{j}, \frac{1}{j}\right)$. Thus,

$$\omega_u \in \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}\left(\frac{1}{j}, \frac{1}{j}\right) \subset \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}\left(\frac{1}{q}, \frac{1}{q}\right) \subset \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho, \varsigma).$$

Therefore $\omega_u \in \mathfrak{i}_{\omega_{u_{n_{\varrho,\varsigma}}}}(\varrho, \varsigma)$ for each $u \in \mathcal{A}$ with $u \geq t_q$. Write $\mathcal{A} = \{k_u : k_u < k_{u+1}\}$. Hence, $(\omega)_{\mathfrak{P}}$ is a Cauchy sequence.

Corollary 2.9. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$, and let (ω_u) be a sequence in Ξ. Then, (ω_u) is a Cauchy sequence in Ξ implies and implied by there is a sequence (q_u) such that $\omega_u = q_u$ for almost all u (λ) and (q_u) is also a Cauchy sequence in Ξ.

Theorem 2.10. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$. If (ω_u) is an λ-statistically convergent sequence in Ξ, then (ω_u) is λ-statistically Cauchy in Ξ.

Proof. The proof follows directly from Theorem 2.6.

Corollary 2.11. Let $(\Xi, \mathfrak{Y}, \delta)$ be a $\mathcal{PGMS}$ and (ω_u) be an λ-statistically convergent sequence in Ξ. Then, there is a subset $\mathfrak{P} = \{k_u : k_u < k_{u+1}\}$ of $\mathbb{N}$ such that $\mathrm{d}^{\lambda}(\mathfrak{P}) = 1$ as well as $(\omega)_{\mathfrak{P}}$ is a Cauchy sequence.

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