Uluslararası İleri Doğa Bilimleri ve Mühendislik Araştırmaları Dergisi
Sayı 7, S. 246-250, 10, 2023

Araştırma Makalesi

International Journal of Advanced Natural Sciences and Engineering Researches
Volume 7, pp. 246-250, 10, 2023

Research Article

https://alls-academy.com/index.php/ijanser
ISSN: 2980-0811

Parameterized Differential Transformation Method for Solving Initial and Boundary Value Problems

Oktay Sh. Mukhtarov [1,2], Merve Yücel*[3] and Kadriye Aydemir[4]

[1]Department of Mathematics/Faculty of Arts and Science, Tokat Gaziosmanpaşa University, Turkey
[2]Institute of Mathematics and Mechanics, Baku, Azerbaijan,
[3]Department of Mathematics/Faculty of Arts and Science, Çorum Hitit University, Turkey
[4]Department of Mathematics/Faculty of Arts and Science, Amasya University, Turkey

*(merve.yucel@outlook.com.tr)

(Received: 05 November 2023, Accepted: 13 November 2023)

(2nd International Conference on Contemporary Academic Research ICCAR 2023, November 4-5, 2023)

ATIF/REFERENCE: Mukhtarov, O. S., Yücel, M. & Aydemir, K. (2023). Parameterized Differential Transformation Method for Solving Initial and Boundary Value Problems. *International Journal of Advanced Natural Sciences and Engineering Researches*, 7(10), 246-250.

Abstract – Recently there has been developed various modifications of some numerical methods of approximating the solution of high order ordinary and partial differential equations. The differential transformation method (DTM) is one of the numerical methods that allows one to find an approximate solution in the case of linear and nonlinear initial and/or boundary value problems for various type of differential equations. This numerical method was first proposed by Zhou, in solving of linear and nonlinear boundary value problems in electrical circuit analysis. The main advantage of DTM is that it can be applied directly to solve nonlinear ordinary and partial differential equations without requiring perturbation or linearization. This study develops a new extension/generalization of DTM called α-parameterized differential transform method (α-PDTM). The α -PDTM differs from classical DTM in the calculation of differential transform coefficients. In this work, we applied the parameterized differential transformation method to solve the following simple but illustrative differential equation.

$$u''(x) + u(x) + 3e^x = 0, \qquad x \in [0,1]$$

together with boundary conditions

$$u(0) = 3, \qquad u'(0) = 0.$$

We also plotted the approximate solution to demonstrate the robustness and efficiency of our own method. The results obtained showed that the proposed new method can become an alternative way to solve boundary value problems of various types. Note that in some concrete values of the auxiliary parameter our own method reduces to the classic DTM.

Keywords – Differential Transform Method, Boundary Conditions, α-Parameterized Differential Transform Method

I. Introduction

Initial and/or boundary value problems (BVPs) for ordinary and partial differential equations of various types arose when modelling various processes in natural science. To understand the nature of such processes, it is necessary to solve model differential equations that satisfy certain initial and/or boundary conditions. In many cases, finding exact solutions to such problems is impossible or very difficult. Therefore, there is growing interest in the development of various numerical methods for solving regular and singular BVP's. Differential Transformation Method (DTM) is one of the simple and effective numerical methods, for solving linear and nonlinear BVP's. Zhou firstly developed the classical version of DTM in 1986 to solve BVPs that arise when modelling electrical circuits [1].

In [3] Ertürk and Momani used the DTM and ADM to get an approximate solutions for fourth - order BVPs. This work also provides a numerical comparison of DTM and ADM-solutions Wazwaz has developed a new generalization of decomposition method for obtaining an approximate solution to a special type high-order BVP's [4]. In some works, the classical DTM was modified so that it can be applied to study not only single-interval BVPs, but also many-interval boundary-value-transmission problems (see, for example [5-7], and references cited thercin)

The main goal of this work is to present a new generalization of the classical DTM for finding numerical or exact solutions to BVPs.

Our own method which we call the parameterized Differential Transmission Method (abbreviated α-PDTM), depends on an auxiliary real parameter α. Note that in the special cases α=0 and α=1 the presented α-PDTM reduces to the classical DTM. We also solved an illustrative BVP for second order ordinary differential equation using α-PDTM to justify the presented method. The results obtained showed that the proposed α-PDTM can be an alternative numerical technique to solve various types of BVPs.

II. Parameterized DTM

Let $s\colon [c,d] \to R$ be a real-valued analytic function and $\alpha \in [0,1]$ be any real parameter.

Definition 2.1. We say the sequence $\big(S_\alpha(c,d)\big)_n(s)$ is the parameterized differential transform of the original function $s(t)$ if

$$\big(S_\alpha(c,d)\big)_n(s) := \alpha\big(S_c(s)\big)_n + (1-\alpha)\big(S_d(s)\big)_n,$$

where

$$\big(S_c(s)\big)_n := \frac{d^n s(c)}{n!} \text{ and } \big(S_d(s)\big)_n := \frac{d^n s(d)}{n!}$$

Definition 2.2. We say the function s(t) is the inverse differential transform if

$$s_\alpha(t) := \sum_{n=0}^{\infty} \big(S_\alpha(c,d)\big)_n(s)\big(t - (\alpha c + (1-\alpha)d)\big)^n \tag{1}$$

provided that the series is convergent. The inverse differential transforms, we will denote by $(S_\alpha^{-1}(c,d)_n)(s)$.

Definition 2.3. The finite sum

$$s_{\alpha,N}(t) := \sum_{n=0}^{N} \big(S_\alpha(c,d)\big)_n(s)\big(t - (\alpha c + (1-\alpha)d)\big)^n \tag{2}$$

that is the N-th partial sum of the inverse differential transformation is said to be an N-th parameterized approximation of the original function.

By Definition 2.1., we can show that the parameterized differential transform has the following properties:

i. $\big(S_\alpha(c,d)\big)_n(\mu s) = \mu(S_\alpha(c,d)(s))_n$

ii. $\big(S_\alpha(c,d)\big)_n(f \pm s) = \big(S_\alpha(c,d)\big)_n(f) \pm \big(S_\alpha(c,d)\big)_n(s)$

iii. $\big(S_\alpha(c,d)\big)_n\left(\frac{d^m s}{dt^m}\right) = \frac{(n+m)!}{n!}\big(S_\alpha(c,d)\big)_n(s)$

III. THe Differential transform method

Let b = b(x) be any analytic function in some around of the point $x = x_0$. Then this function can be expanded in Taylor's series as, given by

$$b(x) = \sum_{l=0}^{\infty} B_{x_0}(l)\,(x-x_0)^l \tag{3}$$

where $B(l)$ is Taylor's coefficient defined by

$$B_{x_0}(l) = \frac{1}{l!}\left[\frac{d^l}{dx^l} b(x)\right]_{x=x_0}, \quad l = 0,1,2,\ldots \tag{4}$$

Definition 3.1 The sequence $B_{x_0}(0)$, $B_{x_0}(1)$, $B_{x_0}(2), \ldots$ is said to be the differential transform of the analytic function $b(x)$, where $B_{x_0}(l)$, $l =$

$0,1,2,\ldots$ is defined by (4). The differential inverse transformation of the sequence $(B_{x_0}(l))$ is defined by (3). Here $b(x)$ is said to be the original function and the sequence $(B_{x_0}(l))$ is said to be the T-transform of $b(x)$.

Let us denote the T- transform of the original function $b(x)$ by $T_{x_0}(b)$, and the differential inverse transform of $(B_{x_0}(l))$ by $T_{x_0}^{-1}(B_{x_0}(l))$. From the definition of the T- transform it follows easily the following properties:

i. $T_{x_0}(a_1 + a_2) = T_{x_0}(a_1) + T_{x_0}(a_2)$
ii. $T_{x_0}(\gamma b) = \gamma T_{x_0}(b)$ for any $\gamma \in R$
iii. If $T_{x_0}(b) = \left(B_{x_0}(l)\right)$, then
$$T_{x_0}\left(\frac{db}{dx}\right) = \left((l+1)B_{x_0}(l+1)\right) \text{ and}$$
$$T_{x_0}\left(\frac{d^2b}{dx^2}\right) = \left((l+1)(l+2)B_{x_0}(l+2)\right)$$
iv. If $T_{x_0}(a) = \left(A_{x_0}(l)\right)$, $T_{x_0}(b) = \left(B_{x_0}(l)\right)$ and $T_{x_0}(ab) = \left(C_{x_0}(l)\right)$, then $C_{x_0}(l) = (A_{x_0}(l) * B_{x_0}(l))$, where $A_{x_0}(l) * B_{x_0}(l)$ is denoted the convolution of the sequences $A_{x_0}(l)$ and $B_{x_0}(l)$.

To find an approximate solution, the differential inverse transform $T_{x_0}^{-1}(B_{x_0}(l))$ can be defined by a finite sum

$$T_{x_0}^{-1}\left(B_{x_0}(l)\right) = \sum_{l=0}^{s} B_{x_0}(l)(x - x_0)^l$$

for sufficiently large s.

IV. Solution of the Problem by Using DTM

Let us consider the differential equation,

$$y''(x) + y(x) + 3e^x = 0, \qquad x \in [0,1]$$

together with the boundary conditions,

$$y(0) = 3, \quad y'(0) = 0.$$

The exact solution for this problem is

$$y(x) = \frac{-3}{2}(-3\cos x + e^x(\cos x)^2 - \sin x + e^x(\sin x)^2$$

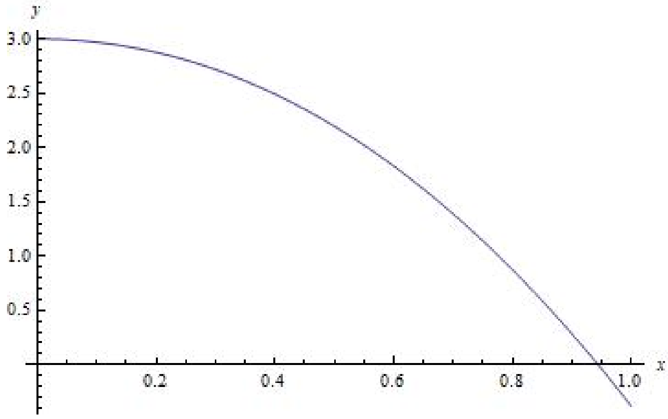

Fig. 1 Exact solution of the problem

Devote by $Y(k)$ the T-transforms of the function $y(x)$ at the end-point $x = 0$. If DTM is applied to the differential equation, we have

$$(l+1)(l+2)Y(l+2) + [Y(l) + \tfrac{3}{l!}] = 0 \quad (5)$$

where $T^{-}(m) = \frac{1}{m!}\left[\frac{d^m}{dx^m}u(x)\right]_{x=x_0}$. The differential inverse transform has the following form:

$$y(x) = \sum_{k=0}^{n} x^k Y(k) = Y(0) + xY(1) + \\ +x^2Y(2) + \cdots + x^8Y(8). \quad (6)$$

The first boundary condition $y(0) = 3$, becomes $Y(0) = 3$, and the second condition $\frac{dy(0)}{dx} = 0$, becomes $Y(1) = 0$.

Now proceed with the iteration using (5); we can calculate the other terms of the T- transform as

$$Y(2) = -3, \quad Y(3) = \frac{-1}{2}, \quad Y(4) = \frac{1}{8}, \quad Y(5) = 0$$
$$Y(6) = \tfrac{-1}{120}, Y(7) = \tfrac{-1}{1680}, \ Y(8) = \tfrac{1}{13440}.$$

If we carry out the iteration up to n = 8, then we have the following approximation solution:

$$y(x) = 3 - 3x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{8}x^4 - \tfrac{1}{120}x^6 - \tfrac{1}{1680}x^7 + \tfrac{1}{13440}x^8. \quad (7)$$

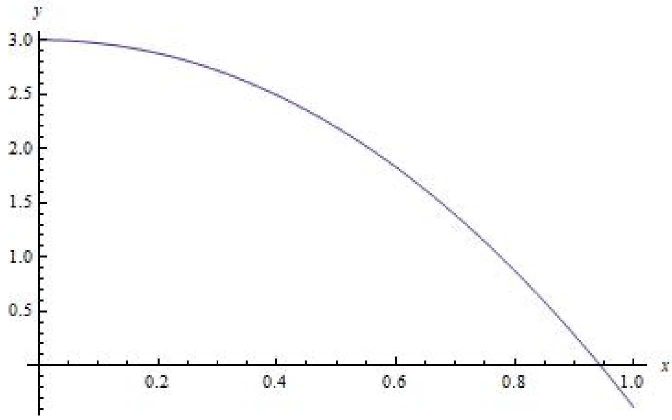

Fig. 2 Graph of the classical DTM solution in the x ∈ [0,1]

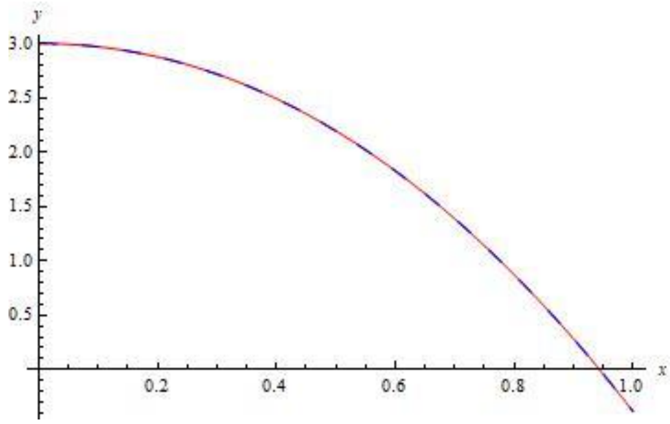

Fig. 3 Graph of the DTM solution (blue line) and the exact solution (red line)

V. APPLICATION OF THE NEW METHOD

Consider the boundary value problem
$y''(x) + y(x) + 3e^{x} = 0, \quad x \in [0,1]$ (8)
subject to the boundary conditions

$$y(0) = 3, \quad y'(0) = 0. \tag{9}$$

If it is applied $\alpha - PDT$ to both sides of (8), then we obtain

$$(S_\alpha(0,1))_{n+2}(y)(n+1)(n+2) = -[\left(S_\alpha(0,1)\right)_n(y)\big) + \frac{3}{n!}]$$

Therefore, from the definition of $\alpha - PDT$, we have

$$y_\alpha(x) = \sum_{n=0}^{\infty}\big(S_\alpha(0,1)\big)_n(y)(x - x_\alpha)^n$$

and

$$y''_\alpha(x) = \sum_{n=0}^{\infty}\big(S_\alpha(0,1)\big)_n(y)n(n-1)(x - x_\alpha)^{n-2}$$

Moreover, for the boundary conditions $y(0) = 3, y'(0) = 0$, we have

$$y_\alpha(0) = \sum_{n=0}^{N}\big(S_\alpha(0,1)\big)_n(y)(\alpha - 1)^n = 3$$

$$y'_\alpha(0) = \sum_{n=0}^{N}\big(S_\alpha(0,1)\big)_n(y)n(\alpha - 1)^{n-1} = 0$$

respectively. Here we write

$$\big(S_\alpha(0,1)\big)_0(y) = \alpha\big(S'_\alpha(0,1)\big)_0 + (1-\alpha)\big(S''_\alpha(0,1)\big)_0$$

…

$$\big(S_\alpha(0,1)\big)_k(y) = \alpha\big(S'_\alpha(0,1)\big)_k + (1-\alpha)\big(S''_\alpha(0,1)\big)_k.$$

Hence, the parameterized series solution $y_\alpha(x)$ is evaluated up to $N = 8$:

$$\begin{aligned} y_\alpha(x) &= \sum_{n=0}^{8}\big(S_\alpha(0,1)\big)_n(y)(x - x_\alpha)^n \\ &= 3\alpha + (1-\alpha)A \\ &\quad + (1-\alpha)B(x - 1 + \alpha) \\ &\quad + \Big[-3\alpha \\ &\quad + (1-\alpha)\frac{-1}{2}(A+3)\Big](x - 1 \\ &\quad + \alpha)^2 + \cdots + [\alpha\frac{1}{13440} \\ &\quad + (1-\alpha)\frac{A}{40320}](x - 1 + \alpha)^8 \end{aligned}$$

We find A and B using boundary conditions.

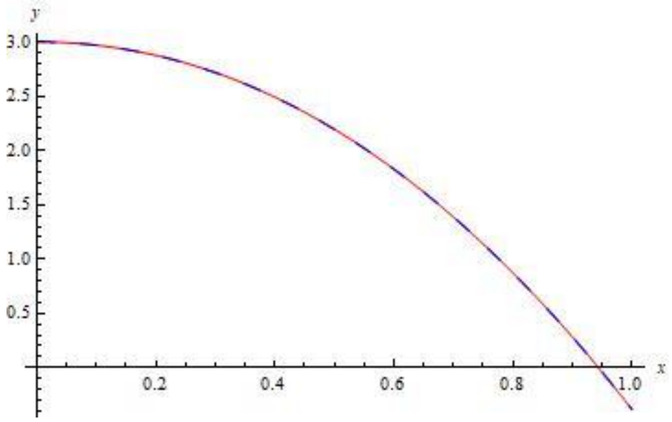

Fig. 4 Graph of the Comparison of the DTM solution (blue line) and the numerical α-parametrized solution for ($\alpha = \frac{1}{2}$) (red line)

VI. CONCLUSION

It is well known that many important initial and/or boundary value problems of mathematical physics cannot be solved analytically. That's why, many numerical or semi - analytical methods, such as Euler Method, Runge Kutta Method, Finite Difference Method, Shooting Method, Adomian Decomposition Method, Homotopy Perturbation Method, Taylor Series Method, etc. have been developed for the calcution of approximate solutions to such problems. One such semi-analytical method is the so-called differential transform method (DTM), which was first used by Zhou in the study of electrical circuit problems. Although this method is based on the Taylor Series Method, as opposed to the Taylor expansion the DTM method provides approximate or exact solutions to boundary value problems without the need to calculate higher derivatives of data functions. Moreover, it can be applied to a fairly wide class of differential equations without linearization, perturbation, or discretization methods that could require large-scale numerical calculations.

This article introduces a new semi-analytical method, so-called parameterized differential transform method to find an approximate solution and in some cases an exact solution in series form. To show the reliability of our method, we solved an illustrative boundary value problem for second order ordinary differential equation.

It is important to note that the proposed α-parameterized DTM is the expansion and generalization of the classical DTM, because in special cases $\alpha = 0$ and $\alpha = 1$, our method is reduced to the classical DTM.

ACKNOWLEDGMENT

The authors declare that there are no conflicts of interest regarding the publication of this article.

REFERENCES

[1] J. K. Zhou, *Differential Transformation and Its Application for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986.

[2] O. Mukhtarov, M. Yücel and K. Aydemir, ‘’Treatment a New Approximation Method and its Justification for Sturm–Liouville Problems’’, *Complexity*, Article ID 8019460, 8 Pages, 2020.

[3] V. S. Ertürk and S. Momani, ‘’Comparing Numerical Methods for Solving Fourth-Order Boundary Value Problems,’’ *Applied Mathematics and Computation*, 188(2), 1963-1968, 2007.

[4] A. M. Wazwaz, ‘’The Numerical Solution of Special Fourth-Order Boundary Value Problems by the Modified Decomposition Method,’’ *International journal of computer mathematics*, 79(3), 345-356, 2002.

[5] D. Arslan, ‘’Approximate Solutions of the Fourth-Order Eigenvalue Problem,’’ *Journal of Advanced Research in Natural and Applied Sciences*, 8(2), 214-221, 2022.

[6] O. Mukhtarov, M. Yücel and K. Aydemir, ‘’A New Generalization of the Differential Transform Method for Solving Boundary Value Problems,’’ *Journal of New Results in Science*, 10(2), 49-58, 2021.

[7] O. Mukhtarov, S. Çavuşoğlu, H. Olğar, Numerical Solution of One Boundary Value Problem Using Finite Difference Method, *Turkish Journal of Mathematics and Computer Science*, 11, 85-89, 2019.