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Research Article

Some results on continuity in a measure-metric space

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***Abstract* –** The symmetric difference of the two intersect sets is subject of many fames and popular books as "Measure theory" from Halmos, "Elementi teorii funkcij i funkcional'nogo analiza" book of Kolmogorov and others. We try to construct one other space parallel with metric space using the symmetric difference. In this space we are studying continuous functions and the convergence of some functional sequences.

Keywords – symmetric difference, metric measure space, weak convergence, weak continuity

I. INTRODUCTION

Let (X, Σ, λ) be a measurable space with Σ a σ-algebra on X and $\lambda < \infty$ one finite Lebesgue measure . We denote $\Pi_\Sigma(X)$ family of all possible subsets of X that are Lebesgue measurable. we consider, as usual, the measure $\lambda : \Pi_\Sigma(X) \to P$ which satisfies the conditions:

D1) Let be $A, B \in \Pi_\Sigma(X)$ and $A,B \neq \varnothing$ .Then $A = B \Leftrightarrow \lambda(A\Delta B) = 0$,

meanwhile $\lambda(\varnothing\Delta\varnothing)=\lambda(\varnothing)=0$.

D2) $\lambda(A\Delta B) = \lambda(B\Delta A)$

D3) $\lambda(A\Delta\varnothing) = \lambda(A)$

D4) $\lambda (A\cup B) \leq \lambda (A) + \lambda (B)$, in the case when $A\cap B=\varnothing$ then $\lambda (A\cup B) =\lambda (A) + \lambda (B)$,

Preposition 1.1.

D5) If $A,B,C \in\Pi_\Sigma (X)$ then $\lambda (A\Delta B) \leq\lambda (A\Delta C) + \lambda (C\Delta B)$.

Preposition 1.2.

(a)If $B \subset A$ then $\lambda(A\Delta B)= \lambda (A\backslash B)= \lambda (A)- \lambda (B)$,

(b) if $B = \{x\}$ then $\lambda (A\Delta\{x\})=\lambda (A\backslash\{x\}\Delta\varnothing)=\lambda (A\backslash\{x\})=\lambda (A)- \lambda (\{x\})$,

(c) If $A = \{y\}$ and $B = \{x\}$ where $x < y$ we consider the inclusion $]-\infty, x] \subset]-\infty, y]$ from it derives that $\lambda (]-\infty, x] \Delta]-\infty, y]= \lambda (]-\infty,y]\backslash]-\infty, x])= \lambda (]-\infty, y])- \lambda (]-\infty, x])= \lambda (]x, y])=y-x$.

Definitions 1.3.

(a) Let A be a subset of Π_Σ (X) and we consider the function $\xi : A\rightarrow\xi(A)= \Pi_\Sigma (A)\cap \Pi_\Sigma (X)$. this is the trace of A in Π_Σ (X). We denote **A** $=\xi(A)$ which is a collection of all subsets which are in $\Pi_\Sigma (A)\cap \Pi_\Sigma (X)$. Among others, $A\in$ **A .**

(b) The open ball with center one subset E of **A** and radius δ is called the set $B(E, \delta) = \{G\in\Pi_\Sigma$ (**A**) : $\lambda(G\Delta E) < \delta\}$. The set e is called *nucleus* of ball.

(c) The set V inner the **A** is called Δ-neighborhood of the set E if it contains the set E together with the ball $B(E, \delta)$. The family of neighborhood of the set e we write $\varsigma(E)$.

(d) The set **G** of all is subset is called Δ-open if it is neighborhood of all its subsets. the complement of an open set Δ-open is called the Δ-closed set**.**

Preposition 1.4. The family $\varsigma(E)$ has the following properties:

1) If $V\in\varsigma(E)$ and $V\subset U$, then $U\in\varsigma(E)$,

2) if V_1 and $V_2 \in\varsigma(E)$, then $V_1\cap V_2\in\varsigma(E)$,

3) $E \subseteq V$, then $V\in\varsigma(E)$,

4) If $U\in\varsigma(E)$ we can find $V\in\varsigma E)$ such that for any $M\subset V$ we have that $U\in\varsigma(M)$

Theorem 1.5.

Family of Δ-open sets T in Π_Σ (X) satisfies these properties

1) $\varnothing$**, X** $\in$T,

2) If **G_1, G_2,... ∈T** then $\bigcup_n G_n \in T$,

3) If **G_1, G_2,... G_n** $\in$T then $\bigcap_{k=1}^{n} G_k \in T$.

We conclude that the space Π_Σ (X) with λ that fulfill the conditions D1)-D5) is the topology space.

Preposition 1.6.

Properties of the family of Δ-closed sets are the same with usual topologies.

Preposition 1.7.

(a) A subset E of the set **G** is called *inner subset* of **G** if we can find one open ball of it that B(E, δ)⊂**G.**

(b) A subset E of the set **G** is called *cluster subset* of **G** if every open ball has non-empty intersection with **G**.

II. CONVERGENCE IN MEASURE-METRIC SPACE

Definition 2.1. We say that the sequence of sets **(G_n)** is Δ- convergent to one given set **E** if for every ε>0 there exists a natural number p∈N such that for n >p we have that $\lambda(\mathbf{G_n \Delta E}) < \varepsilon$.

The set E is called Δ -limes of sequence and we denote $\mathbf{G_n} \xrightarrow{\Delta} E$.

Preposition 2.2. The Δ -limes of the set sequence is unique.

Example 2.3. Let (G_n) be a monotonous non -decreasing set sequence $G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots$ and $E = \bigcup_{n=1}^{\infty} G_n$ then $\mathbf{G_n} \xrightarrow{\Delta} \mathbf{E}$.

Definition 2.4. Let G be a set of Π_Σ (X), we say that **G** is Δ-bounded if there exists a ball B(E, r) for one fixed E such that **G** ⊂ B(E, r) such that maybe r is very large but finite.

Preposition 2.5. If the sequence **(G_n)** of the sets is Δ-convergent to the set E then it is bounded.

Definition 2.6. The sequence **(G_n)** of the sets is Δ-Cauchy if for every ε > 0 there exists a natural number p ∈ N such that for the indexes m, n > p we have $\lambda(\mathbf{G_n \Delta G_m}) < \varepsilon$.

Preposition 2.7.

(a) The convergent sequence of the sets is sequence Δ-Cauchy.

(b) The Δ-Cauchy sequence are Δ-bounded.

Definitions 2.8.

(a) Let f be one function of the sets such that f : Π_Σ (X)→ Π_Σ (X) on a measurable space (X, Σ, λ). The set E ∈Π_Σ (X) fixed set of the function f if and only if f(E) = E. In our case we can writhe that μ(f(E)ΔE)=0.

(b) the set function f is called contract in X if and only if for every two sets E, F ∈Π_Σ (X) it holds

$\mu(f(E)\Delta f(F)) \leq q.\mu(E\Delta F)$

Where 0 < q < 1 and μ numerable additive.

Theorem 2.9. (Banach theorem)

Let X be a measure-metric space and $X \neq \emptyset$. If X is a Δ-complete space and $f : \Pi_\Sigma(X) \to \Pi_\Sigma(X)$ is a retract there exists a fixed set E such that F(E) = E and this is unique.

Theorem 2.10. A measure-metric space X is Δ -complete if and only if for every sequence $B_n(A_n, r_n)$ of balls inserted inside each other and their radius go to zero when $n \to \infty$ there exists a set $E \in \Pi_\Sigma(X)$ noon empty such that is common for all the balls of the sequence.

III. CONTINUOUS FUNCTION

Let X, Y by two sets non empty and their measure - metric spaces $(X, \Sigma, \lambda, \Pi_\Sigma(X))$ and $(Y, \Gamma, \mu, \Pi_\Gamma(Y))$, where Σ and Γ are respectively σ-algebras and λ, μ finite measure.

Definition 3.1. The set function $f : \Pi_\Sigma(X) \to \Pi_\Gamma(Y)$ is called Δ-continuous on subset E if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every set $G \in \Pi_\Sigma(X)$ for which $\lambda(G\Delta E) < \delta$ we have $\mu(f(G)\Delta f(E)) < \varepsilon$.

Noting $B(E, \delta) = \{G \in \Pi_\Sigma(X) : \lambda(G\Delta E) < \delta)$ open ball on X and $B(f(E), \varepsilon) = \{ f(G) \in \Pi_\Gamma(Y) : \mu(f(G)\Delta f(E)) < \varepsilon$. we can geometrize the definition of continuous of function.

The set function $f : \Pi_\Sigma(X) \to \Pi_\Gamma(Y)$ is called Δ-continuous on subset E if for every Δ-open ball $B(f(E), \varepsilon)$ of $\Pi_\Gamma(Y)$ there exists Δ -open ball $B(E, \delta)$ on $\Pi_\Sigma(X)$ such that for every $G \in B(E, \delta)$ we have that $f(G) \in B(f(E), \varepsilon)$. (or $f(B(E, \delta)) \subset B(f(E), \varepsilon)$).

Definition 3.2 (by sequences) The set function $f : \Pi_\Sigma(X) \to \Pi_\Gamma(Y)$ is called Δ-continuous on subset E if and only if for every sequence $G_1, G_2, ..., G_n, ...$ Δ_λ -convergent to E, the respective sequence of function values $f(G_1), f(G_2), ..., f(G_n), ...$ is Δ_μ - convergent to f(E).

Preposition 3.3. The two definitions of Δ - continuous of functions are equivalent.

As a special case we say that measure μ is Δ-continuous by λ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every two subsets $G, F \in \Pi_\Sigma(X)$ such that $\lambda(G\Delta F) < \delta$ we have $|\mu(G)-\mu(F)| < \varepsilon$ (or

$\mu(G\Delta F) < \varepsilon$).

Preposition 3.4. Measure-metric μ is absolutely continuous.

Preposition 3.5. Let f be the function $f : \Pi_\Sigma(P) \to \Pi_\Gamma(P)$ where $(X, \Sigma, \lambda, \Pi_\Sigma(P))$ and $(Y, \Gamma, \mu, \Pi_\Gamma(P))$ measure-metric spaces. If the function f is Δ-continuous in one $\mathbf{E} \in \Pi_\Sigma(P)$ then if is uniformly continuous on that set.

Preposition 3.6. The function $f : \Pi_\Sigma(P^n) \to \Pi_\Gamma(P^n)$ is Δ-continuous in one point $x_0 \in P^n$ if and only if it is continuous as the function $f : P^n \to P^n$ at this point.

We proof this when n =2.

Preposition 3.7. The function f : ΠΣ(X) →ΠΓ(Y) is Δ-continuous in one set H ⊂ **Y** if and only if the pre-image of one Δ-neighborhood of the set H is a Δ-neighborhood of the set G= f^{-1}(H).

Corollary 3.8. The above function is Δ-continuous on ΠΣ(X) if and only if the pre-image of every Δ-open set (or Δ-closed set) is Δ-open(or, respectively Δ-closed) in ΠΓ(Y).

IV. CONVERGENCE OF FUNCTIONAL SEQUENCES

Definitions 4.1. Let $(f_n)_{n\in N}$ a sequence of functions such that f : ΠΣ(X) →ΠΓ(Y) and G ∈ **A**∈ ΠΣ(X).

(a) We say that the sequence $(f_n)_{n\in N}$ is discretely convergent to element G ∈ **A** if for every ε > 0 there exists p(ε, G)∈ N such that for n > p(ε, G) we have $\mu(f_n(G)\Delta f(G)) < \varepsilon$.

(b) We say that the sequence $(f_n)_{n\in N}$ is uniformly convergent to element G ∈ **A** if for every ε > 0 there exists p(ε)∈ N such that for n > p(ε) we have $\mu(f_n(G)\Delta f(G)) < \varepsilon$.

(c) We say that the sequence $(f_n)_{n\in N}$ is Δ-δ convergent to element G ∈ **A** if for every ε > 0 there exists p(ε, G)∈ N such that for n > p(ε, G) and H ⊂ B(G, δ) we have $\mu(f_n(H)\Delta f(H)) < \varepsilon$.

(d) We say that the sequence $(f_n)_{n\in N}$ is Δ-δ_a convergent to element G ∈ **A** if for every ε > 0 there exists p(ε, G)∈ N such that for n > p(ε, G) and H ⊂ B(G, δ) we have $\mu(f_n(H)\Delta f(G)) < \varepsilon$.

(e) We say that the sequence $(f_n)_{n\in N}$ is Δ-α convergent at set G if for every sequence $G_n \xrightarrow{\Delta} G$ we have that $f_n(G_n) \xrightarrow{\Delta} f(G)$.

(f) We say that the sequence $(f_n)_{n\in N}$ is exhaustive on A ∈ΠΣ(X) if for every ε > 0 there exists p(ε)∈ N such that for n > p(ε) and H ⊂ B(G, δ) we have $\mu(f_n(H)\Delta f_n(G)) < \varepsilon$.

From the Definitions we see that Δ- uniform convergence derive the Δ- discretely convergence and Δ-δ convergence.

Preposition 4.2. Let f, f_n be the functions that f, f_n : ΠΣ(X) →ΠΓ(Y). If the sequence $(f_n)_{n\in N}$ Δ-uniformly convergent with Δ-continuous terms then it is convergent to the Δ-continuous function f.

Preposition 4.3. If the above sequence $(f_n)_{n\in N}$ is Δ - uniformly convergent to function f on **A** ∈ΠΣ(X) the sequence is Δ - exhaustive on **A**.

Preposition 4.4. If the sequence $(f_n)_{n\in N}$ is Δ - uniformly convergent to function f on **A** ∈ΠΣ(X) the sequence f_n is Δ-α convergent.

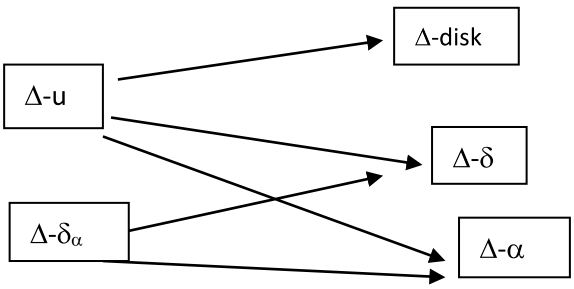
Preposition 4.5. If the above sequence $(f_n)_{n\in N}$ is Δ-δ_a - convergent in one set **A** ∈ΠΣ(X) then it is Δ-α - convergent.

Preposition 4.6. Let f, f_n be the functions that f, f_n : ΠΣ(X) →ΠΓ(Y). If the sequence f_n is Δ-δ-convergent on the set **A** ∈ΠΣ(X) then

(i) It is exhaustive on **A**,

(ii) f is Δ- contionuous on **A**

(iii) it is Δ-δ- convergent.



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