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Research Article

An Application of Conformable Laplace Decomposition Method to Fractional Kaup-Kupershmidt Equation

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Abstract – In the field of nonlinear wave dynamics, this comprehensive study explores advanced computational methodologies for solving complex evolution equations. Utilizing the conformable Laplace decomposition method, we present a sophisticated mathematical framework for examining complex wave transformation mechanisms. By integrating fractional calculus principles with innovative decomposition techniques, our research reveals profound insights into the behavior of nonlinear evolutionary systems. The study demonstrates how computational strategies can effectively decode wave propagation characteristics, offering researchers a powerful tool for understanding complex dynamic processes across various scientific domains.

Keywords – Kaup-Kupershmidt Equation; Conformable Derivative; Adomian Decomposition Method.

I. INTRODUCTION

The concept of fractional calculus, encompassing non-integer order differentiation and integration, has roots that can be traced back to the early foundations of traditional integer-order calculus. While much of the theoretical development related to fractional calculus was completed by the end of the 19th century, it is only within the last century that significant advancements in its engineering and scientific applications have emerged [1,2]. In certain cases, the computational methods have been adapted to better align with physical phenomena [3]. The application of fractional derivatives in modeling real-world problems has become increasingly widespread in recent decades. Notable examples include the use of fractional calculus in seismic analysis, fluid dynamic models incorporating fractional derivatives, and the characterization of viscoelastic material properties, among others.
In the literature on fractional analysis, various definitions have been proposed to generalize the concept of differentiation to fractional orders, including the Riemann–Liouville, Grünwald–Letnikov, Caputo, and Generalized Functions approaches. One of the more recent contributions to this field is the introduction of the conformable derivative in 2014, which provides an alternative framework for understanding fractional differentiation, building upon the foundational theories of earlier methods. This derivative has gained

attention due to its distinct properties and applicability to certain types of fractional differential equations, offering researchers a novel perspective in the study of complex systems governed by non-integer order dynamics [4,5].
Nonlinear differential equations, particularly those of fractional order, are addressed through various sophisticated numerical and analytical methods. Researchers employ multiple advanced techniques for approximating solutions, including the tanh method, Padé approximation, Adomian decomposition method, and variational iteration method, conformable Laplace decomposition method (CLDM), etc. [6-9]. CLDM is a hybrid method obtained by combining the conformable Laplace transform and the Adomian decomposition method. It represents a particularly effective approach for solving fractional-order nonlinear differential equations. In this context, we will demonstrate an approximate solution to the Kupper-Schmidt equation utilizing the CLDM methodology [10].

II. MATERIALS AND METHOD

To capture the refined mathematical behavior of fractional-order derivatives, we need to give the following definition and theorems.

Definition 1.1.

Let h be a function with domain $[0,\infty) \to$ and range $\mathbb{R}$. For $t > 0$ and $\theta \in (0,1)$, the conformable θ- order fractional derivative of h is given by:

$$D_t^{\theta}(h)(t) = \lim_{a\to 0} \frac{h(t+at^{1-\theta})-h(t)}{a}\,. \tag{1}$$

Additionally, $D_t^{\theta}(h)(0) = \lim_{t\to 0} D_t^{\theta}(h)(t)$ if h is θ -differentiable in$(0,p)$ for some $p > 0$ and if $\lim\,(t \to 0)$, $D_t^{\theta}(h)(t)$ exists, as established by [5].

Theorem 1.1.

Assume s, h be η-differentiable functions at some point $t > 0$ and $\eta \in (0,1]$. Then [5]

1. $D_t^{\eta}(hp + sk) = p\,D_t^{\eta}(h) + kD_t^{\eta}(s)$ for any real k, p constants.
2. $D_t^{\eta}(d) = 0$ for any constant function $s(t) = d$
3. $D_t^{\eta}(hs) = D_t^{\eta}(h)s + h\,D_t^{\eta}(s)$
4. $D_t^{\eta}\left(\frac{h}{s}\right) = \frac{D_t^{\eta}(h)s - h\,D_t^{\eta}(s)}{s^2}$
5. $D_t^{\eta}(t^m) = mt^{m-\eta}$ for any real m
6. $D_t^{\eta}(hog) = h'(s(t))D_t^{\eta}(s)(t)$ when h is differentiable at $s(t)$

Definition 1.2.

Consider a function w: $[d,\infty) \to \mathbb{R}$ with $a \in \mathbb{R}$ and $0 < \eta \leq 1$. Then η-order conformable For the function w, Laplace transform is defined as [11]

$$\mathcal{L}_{\eta}^{d}[w(t)](s) = \int_d^{\infty} e^{-s\frac{(t-d)^{\eta}}{\eta}}\, w(t)d_{\eta}(t,d) = \int_d^{\infty} e^{-s\frac{(t-d)^{\eta}}{\eta}} w(t)(t-d)^{\eta-1}dt \tag{2}$$

Method Algorithm

Having established the theoretical context for fractional-order nonlinear differential equations, we now turn our attention to the detailed algorithmic framework of the CLDM.

Consider the following fractional PDE

$$D_t^r\, \upsilon + S(\upsilon) + N(\upsilon) = p(x,t) \qquad 0 < r \leq 1, x > 0,\ t > 0 \tag{3}$$

with the initial values

$$\upsilon(x,0) = m(x) \tag{4}$$

where D_t^r linear conformable sense derivative operators, S represents the remaining linear terms, N represents the nonlinear terms and m represents the initial vaue.

If the CLT is applied to Eqn.(5), followed by the application of the inverse CLT with initial values

$$\upsilon = \mathcal{L}_r{}^{-1}[\tfrac{1}{s}(\upsilon(x,0) + \mathcal{L}_r[p])] - \mathcal{L}_\eta{}^{-1}[\tfrac{1}{s}\big(S(\upsilon)\big)] - \mathcal{L}_r{}^{-1}[\tfrac{1}{s}\big(N(\upsilon)\big)] \tag{5}$$

is obtained.

According to the Adomian Decomposition Method (ADM); the solution υ, nonlinear terms $N(x,t)$ and Adomian polynomials B_k are respectively represented as [12-14] :

$$\upsilon(x,t) = \sum_{n=0}^{\infty} \upsilon_n \tag{6}$$

$$N\big(\upsilon(x,t)\big) = \sum_{n=0}^{\infty} B_n \tag{7}$$

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\upsilon_0 + \sum_{i=1}^{n} \lambda^i \upsilon_i]_{\lambda=0} \quad , \quad n = 0,1,2,\ldots \tag{8}$$

Substituting these series representations into the Eqn. (5), an iterative algorithm is obtained as :

$$\upsilon_0 = \mathcal{L}_r{}^{-1}[\tfrac{1}{s}(\upsilon(x,0) + \mathcal{L}_r[p])] \tag{9}$$

$$\upsilon_{n+1} = -\mathcal{L}_r{}^{-1}[\tfrac{1}{s}\big(H(\upsilon_n)\big)] - \mathcal{L}_\eta{}^{-1}[\tfrac{1}{s}\big(P(\upsilon_n)\big) \tag{10}$$

By calculating the desired number of υ_n terms, approximate analytical solutions for $\upsilon(x,t)$ can be found from Eqn. (9) and Eqn. (10).

The key points here are the use of the conformable Laplace transform to convert the differential equation to an algebraic form, the simplification using the differential property, the representation of the solution as an infinite series using Adomian Polynomials, and the development of the iterative algorithm to calculate the solution components.

III. RESULTS AND DISCUSSION

In this section, we demonstrate the efficiancy of the CLDM by implementing a comprehensive numerical investigation. We will systematically apply the proposed methodology to a representative mathematical model, providing visual representations and comparative analyses through meticulously constructed graphical illustrations.

Example 2.1.

Consider the given non-linear conformable fractional Fokker Planck PDE below [15]

$$\frac{\partial^r \upsilon}{\partial t^\eta} + 45\upsilon^2\frac{\partial \upsilon}{\partial x} - 15\sigma\frac{\partial \upsilon}{\partial x}\frac{\partial^2 \upsilon}{\partial x^2} - 15\sigma\upsilon\frac{\partial^3 \upsilon}{\partial x^3} + \frac{\partial^5 \upsilon}{\partial x^5} = 0 \qquad , \quad 0 < r \leq 1 \tag{11}$$

initial value

$$\upsilon(x,0) = \frac{1}{4}c^2\lambda^2 sech^2(\frac{\lambda c x}{2}) + \frac{1}{12}c^2\lambda^2 \tag{12}$$

with the exact solution of

$$\upsilon(x,t) = \frac{1}{4}c^2\lambda^2 sech^2(\frac{\lambda}{2}(\frac{\varepsilon t^r}{r} + cx)) + \frac{1}{12}c^2\lambda^2 \tag{13}$$

If the CLT is applied to Eqn.(14), followed by the application of the inverse CLT with the initial values

$$\upsilon = \mathcal{L}_r{}^{-1}\left[\frac{1}{s}\left(\frac{1}{4}c^2\lambda^2 sech^2\left(\frac{\lambda c x}{2}\right)+\frac{1}{12}c^2\lambda^2\right)\right]-\mathcal{L}_\eta{}^{-1}\left[\frac{1}{s}\left(\frac{\partial^5 \upsilon}{\partial x^5}\right)\right]$$
$$-\mathcal{L}_r{}^{-1}[\frac{1}{s}\left(45\upsilon^2\frac{\partial \upsilon}{\partial x}-15\sigma\frac{\partial \upsilon}{\partial x}\frac{\partial^2 \upsilon}{\partial x^2}-15\sigma\upsilon\frac{\partial^3 \upsilon}{\partial x^3}\right)] \quad (14)$$

is obtained.

Substituting the series representations Eqn.(6)-Eqn.(8) into Eqn.(14), the following iterative algorithm is obtained.

$$\upsilon_0 = \frac{1}{4}c^2\lambda^2 sech^2\left(\frac{\lambda c x}{2}\right)+\frac{1}{12}c^2\lambda^2 \quad (15)$$
$$\upsilon_{n+1} = -\frac{675}{8}c^7\lambda^7 tanh^6\left(\frac{\lambda c x}{2}\right)+\cdots-\frac{45}{16r^n}c^7\lambda^7 t^{(n+1)r} \quad (16)$$

Hence the 4-step approximate CLDM solution of $\upsilon(x,t)$ is obtained as

$$\begin{aligned}\upsilon(x,t)_{CLDM_4} &= \frac{1}{4}c^2\lambda^2 sech^2\left(\frac{\lambda c x}{2}\right)+\frac{1}{12}c^2\lambda^2\\ &-\frac{675}{8r}c^7\lambda^7 t^r tanh^6\left(\frac{\lambda c x}{2}\right)+\cdots-\frac{45}{16r}c^7\lambda^7 t^r\\ &+\frac{315}{512r^2}c^{12}\lambda^2 tanh^{12} t^{2r}\left(\frac{\lambda c x}{2}\right)-\cdots-\frac{1113}{4096r^2}c^{12}\lambda^{12}t^{2r}\\ &+\frac{60810750}{6144r^3}c^{17}\lambda^{17}t^{3r} tanh^{17}\left(\frac{\lambda c x}{2}\right)-\cdots-\frac{66825}{32768r^3}c^{17}\lambda^{17}t^{3r}\end{aligned} \quad (17)$$

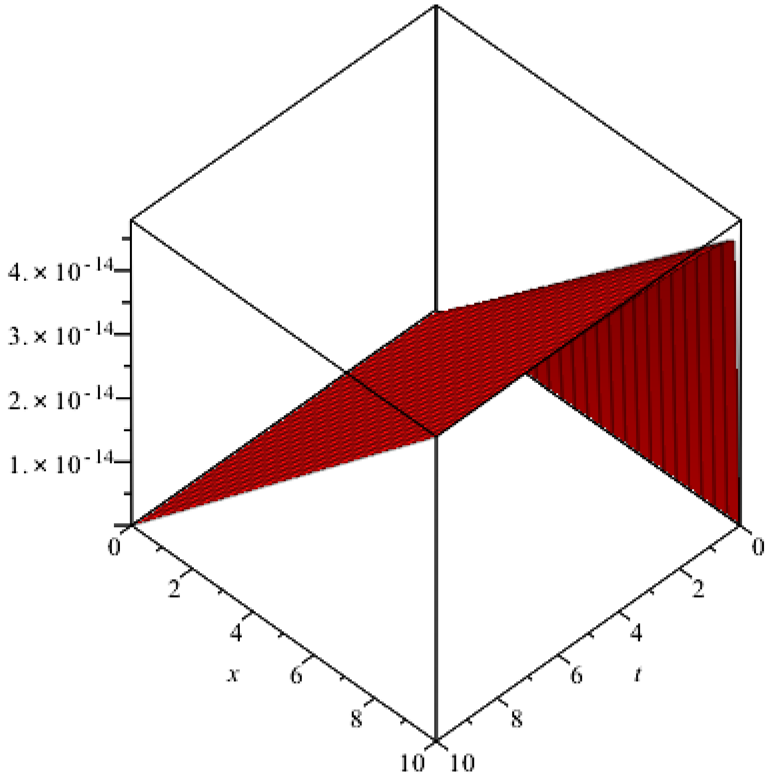

Figure 1. Error between 4-step CLDM solution and exact solution of $\upsilon(x,t)$ when $r = 0.2$ with $0 \leq t \leq 10$ and $0 \leq x \leq 10$

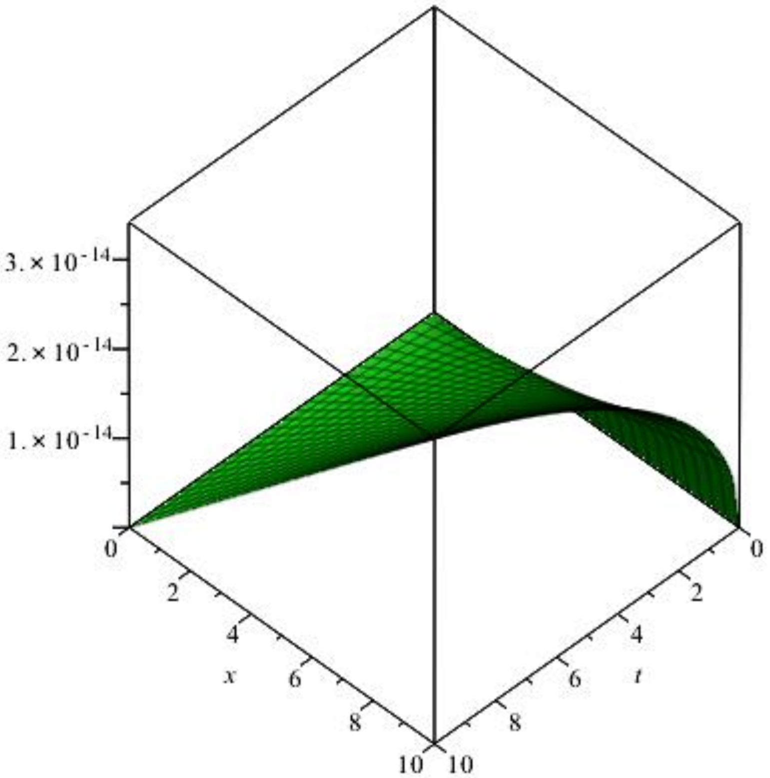

Figure 2. Error between 4-step CLDM solution and exact solution of $\upsilon(x,t)$ when $r = 0.7$ with $0 \leq t \leq 10$ and $0 \leq x \leq 10$

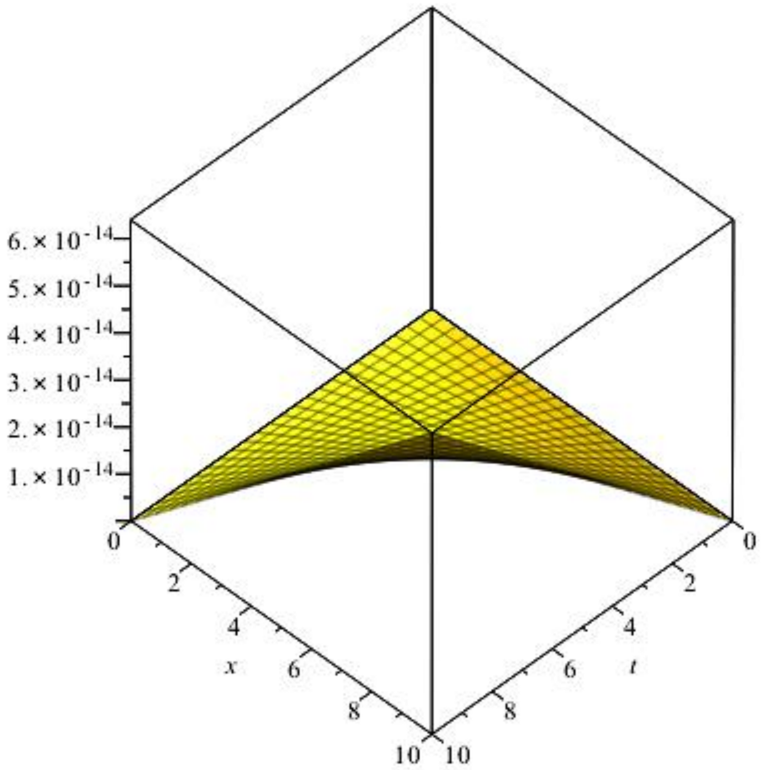

Figure 3. Error between 4-step CLDM solution and exact solution of $v(x,t)$ when $r = 1.0$ with $0 \leq t \leq 10$ and $0 \leq x \leq 10$

In Figure1, Figure2 and Figure3, the $3D$ surface plots show the error difference between the exact solution and the $CLDM$ approximation $v(x,t) - v(x,t)_{CLDM}$ are given for different r values.

- For $r = 0.2$: Shows minimal error concentration near the boundaries
- For $r = 0.7$: Demonstrates improved accuracy across the domain
- For $r = 1.0$: Exhibits the best accuracy, confirming the method's consistency with classical calculus

Here it is aimed to show how the change in derivative order affects the solution. The error magnitudes remain consistently small (order of 10^{-14}) across all cases, demonstrating remarkable accuracy of the CLDM approach for both fractional and integer-order derivatives. The smooth error surfaces indicate stable numerical behavior of the method throughout the solution domain

IV. CONCLUSION

In this study, we investigated the Kaup-Kupershmidt equation, which plays a crucial role in modeling nonlinear wave phenomena in fluid dynamics and plasma physics. The CLDM method was successfully applied to solve this equation, demonstrating remarkable accuracy with just a few terms in the series solution. Our numerical results show that the method is highly efficient, with error margins in the order of 10^{-14}. The comparison between exact and approximate solutions for different values of α validates the reliability of our approach, particularly showing that the method maintains its effectiveness as α approaches both fractional and integer orders. The three-dimensional visualizations and error analysis confirm that CLDM provides a powerful tool for analyzing such complex equations. This work contributes significantly to the field of fractional differential equations and offers a promising framework for solving similar nonlinear problems in mathematical physics and engineering applications.

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