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***Research Article***

# Level Sequence of A Class of Trees

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***Abstract*** **–** In this note, we obtain a level based sequence of rooted caterpillars whose vertices are located on the spines have equal number of attached leaves. We obtain the general formula of the level index of the mentioned caterpillar and give a characterization of the level index of the caterpillar by the solution of a difference equation.

*Keywords – Level Index, Sequence, Distance, Tree, Caterpillar*

## I. INTRODUCTION

Integer sequences which are based on the levels of vertices were reported by Flajolet and Prodinger [1]. They gave the explicit asymptotic expressions of the level number sequences of the corresponding binary trees. A different characterization of level number sequences of $d$-ary trees was attained by Tangora [2]. The author investigated the level number sequences in term of algebraic topology.

Analysis of rooted trees by the levels of the vertices were continued by the authors Balaji and Mahmoud [3]. They introduced two distance based invariants which one of them is level index. This index can be computed by the sum of differences of levels of corresponding vertices. The level of a vertex equals to the distance from the root vertex to it.

Level polynomial which counts the levels of vertices was defined in the paper [4] and level matrix whose components are equal to the difference of levels of two any vertices was defined in the paper [5].

In this note, we obtain a level based sequence of rooted caterpillars whose vertices are located on the spines have equal number of attached leaves. We obtain the general formula of the level index of the mentioned caterpillar and give a characterization of the level index of the caterpillar by the solution of a difference equation.

## II. MATERIALS AND METHOD

**Definition 2.1.** Balaji and Mahmoud introduced two distance based topological indices for rooted trees [3]. The first one is called level index and level index of a tree is denoted by $L(T)$. Level index of a tree $T$ is computed by the following equation

$$L(T) = \sum_{1 \le i < j \le n} \left| D_j(T) - D_i(T) \right|$$

such that $D_i(T)$ and $D_j(T)$ showing the vertices at distances $i$ and $j$ from the root vertex of the tree $T$.

We can write it

$$L(T) = \sum_{1 \le i < j \le n} |l_j(T) - l_i(T)|$$

**Definition 2.2.** Level index of any rooted tree can be computed by the following equation

$$L(T) = \sum_{i=0}^{h} \sum_{j=0}^{h-j} j N_i N_{i+j}$$

such that $N_i$ shows the number of vertex of the level $i$.

## III. RESULTS

It means that level index equals tos um of the differences between the levels of vertices with respect to a reference vertex. The reference vertex is called root vertex for the caterpillar tree. Assume that a path $P_n$ is rooted at a vertex $v_0$ as $v_0, \ldots, v_{n-1}$and $k$ vertices are attached to each vertex of the path $P_n$. By this way we obtain that there is one vertex at $l_0$, there are $k+1$ vertices at $l_i$ $(1 \le i \le n-1)$ and there are $k$ vertices at $l_n$. The mentioned rooted caterpillar is denoted by $L_n$. If the last vertex $v_{n-1}$ and its attached $k$ vertices are removed from $L_n$, we will attain the rooted caterpillar $L_{n-1}$ and this goes on to rooted star $L_1$ with the root vertex $v_0$ and $k$ leaves of $v_0$.

We can compute the initial terms of $L_n$ as follows.

$$L(L_1) = k$$

$$L(L_2) = 2k + (k+1)^2$$

$$L(L_3) = 3k + 4(k+1)^2.$$

**Theorem 3.1.** Level index of the caterpillar $L_n$ equals to following equation

$$L(L_n) = nk + \binom{n+1}{3}(k+1)^2$$

**Proof.** By Definition 2.2 we can obtain the level index of $L_n$

$$L(L_n) = nk + (n-1)[1(k+1) + (k+1)k]$$

$$+(n-2)[1.(k+1) + (k+1)(k+1) + (k+1)k]$$

$$+ \cdots + 1[1.(k+1) + (n-2)(k+1)^2 + (k+1)k]$$

$$L(L_n) = nk + (k+1)^2[(n-1).1 + (n-2).2 + \cdots + 1.(n-1)]$$

$$= nk + (k+1)^2 \sum_{i=1}^{n-1} i(n-i)$$

$$= nk + (k+1)^2\left[\frac{n^2(n-1)}{2} - \frac{(n-1)n(2n-1)}{6}\right]$$

$$= nk + \binom{n+1}{3}(k+1)^2$$

**Corollary 3.2.** If $k$ is choosen 0, we will obtain the path $P_n$ with the level index $L(P_n) = \binom{n+1}{3}$.

Since the level index is computed by the differences of levels of vertices and $L_{i-1}$ can be obtained from $L_i$ by the same removing method, the level index of $L(L_{n+1})$ equals to

$$L(L_{n+1}) = (n+1)k + \binom{n+2}{3}(k+1)^2$$

Now we take the difference of $L(L_{n+1})$ and $L(L_n)$ as follows

$$L(L_{n+1}) - L(L_n) = k + \binom{n+1}{2}(k+1)^2 \quad (*)$$

$$L(L_{n+1}) - L(L_n) = k + \frac{n^2+n}{2}(k+1)^2$$

Since the $(*)$ equation is a difference equation, we have to compute homogeneous and private solution of $(*)$ as in the following equation.

**Theorem 3.3.** The complete solution of the $(*)$ equation equals to

$$S_c\big(L(L_n)\big) = \frac{(k+1)^2}{6}n^3 - \frac{k^2-4k+1}{6}n + c$$

**Proof.** We look for the homogeneous solution

$$L(L_{n+1}) - L(L_n) = k + \frac{n^2+n}{2}(k+1)^2$$

Since $x-1=0$, $x=1$ and homogeneous solution $S_h$ equals to $S_h = c.1^n = c$

The private solution of the equation $(*)$ is computed by the following equation

$$S_p[L(L_n)] = (An^2 + Bn + C)n$$

$$= An^3 + Bn^2 + Cn$$

and

$$S_p[L(L_{n+1})] = A(n+1)^3 + B(n+1)^2 + C(n+1).$$

Therefore we can write the difference of consecutive terms as

$$S_p[L(L_{n+1})] - S_p[L(L_n)] = k + \frac{n^2+n}{2}(k+1)^2$$

$$A(n+1)^3 + B(n+1)^2 + C(n+1) - An^3 - Bn^2 - Cn = k + \frac{n^2+n}{2}(k+1)^2$$

$$A(3n^2+3n+1)+B(2n+1)+C=\frac{(k+1)^2}{2}n^2+\frac{(k+1)^2}{2}n+k$$

$$3An^2+(3A+2B)n+A+B+C=\frac{(k+1)^2}{2}n^2+\frac{(k+1)^2}{2}n+k$$

We find the values of $A, B$ and $C$ by the following equations

$$3A=\frac{(k+1)^2}{2}$$

$$3A+2B=\frac{(k+1)^2}{2}$$

$$A+B+C=k$$

$$A=\frac{(k+1)^2}{6}, B=0, C=-\frac{k^2-4k+1}{6}$$

By this way we obtain the private solution such that

$$S_p[L(L_n)]=\frac{(k+1)^2}{6}n^3-\frac{k^2-4k+1}{6}n$$

and the complete solution

$$S_c=S_p+S_h=\frac{(k+1)^2}{6}n^3-\frac{k^2-4k+1}{6}n+c$$

A different characterization of $L(L_n)$ can be made by the following difference equation. We can take the difference of $L(L_{n+1}), L(L_n)$ and $L(L_n)-L(L_{n-1})$. If the difference of following two equations is taken, it will be obtained the difference equation $(**)$ with three consecutive terms.

$$L(L_{n+1})-L(L_n)=k+\binom{n+1}{2}(k+1)^2$$

$$L(L_n)-L(L_{n-1})=k+\binom{n}{2}(k+1)^2$$

$$L(L_{n+1})-2L(L_n)+L(L_{n-1})=n(k+1)^2 \; (**)$$

## IV. CONCLUSION

We obtained the level sequence of a class of rooted caterpillar, computed the level index of the mentioned trees and characterized the difference of two consecutive terms of this sequence.

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