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Research Article

A Revıew of Mathematical Conjectures: Exploring Engaging Topics for University Mathematics Students

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Abstract **–**Throughout history, humanity has been driven by an innate curiosity to explore beyond established boundaries, particularly evident in scientific and mathematical pursuits. The realm of mathematics has seen numerous conjectures spanning ancient times to the present day, encompassing various mathematical domains. These conjectures, some evolving into theorems upon proof, others being refuted and replaced, and a few remaining yet unresolved, form a significant facet of intellectual exploration. They captivate not only professional mathematicians but also enthusiasts, contributing to the evolution of mathematics. Mathematical conjectures are statements that have not yet been proven to be true or false. Typically created from observed patterns, these conjectures often originate from seemingly simple propositions. Presently, advancements in computer programming have substantially contributed to and aided in proving wrong by finding some counterexamples or confirming the conjectures for very large numbers. Python, in particular, facilitates the verification of conjectures for larger numbers, the identification of patterns and formulas, confirming conjectures or helping in finding counterexamples leading to rejection, as well as refining existing ones or generating new ones. The article aims to present several famous math conjectures, predominantly in number theory, and emphasize the importance and use of working with students for a more interesting class. Notable conjectures include Euclid's perfect number conjecture, Fermat's number conjecture, Collatz's conjecture, Landau's conjecture, Mersenne's prime conjecture, and more.

Keywords: Conjecture, Mathematics, Python, Coding, Programming

I. INTRODUCTION

The origins of mathematical conjectures can be traced back to ancient civilizations, where scholars and mathematicians began to observe recurring patterns in numbers, shapes, and mathematical relationships [1]. These early conjectures, formulated amidst numerical patterns, set the stage for the evolution of mathematical thought. Numerous conjectures have emerged as sparks igniting mathematical curiosity [2]. Some of the most renowned conjectures have evolved into theorems, solidifying their status as proven mathematical truths. Yet, others remain unresolved, challenging the intellect and perseverance of mathematicians for generations [3]. The transition

of conjectures into theorems represents a triumph of rigorous proofs, unlocking new avenues of mathematical understanding and enriching the tapestry of mathematical knowledge [4].
Several most notable conjectures are:

- Euclid's Perfect Number Conjecture.

Perfect number are mentioned in Euclid’s book, “The Elements”. He even produced a formula to generate the perfect numbers
$N = (2^{p-1})(2^p - 1)$, with p and $2^p - 1$ being prime.
Euclid and his successors assumed evidently that all perfect numbers were of the form he provided, so all were even.
Among many mathematiciens, Pierre de Fermat, Leonhard Euler made significant contributions to the study of perfect numbers.
The conjecture that all perfect numbers are even remains open till today, and the use of computer programming for the calculation of very large perfect numbers has not produced any counterexample to reject the conjecure [5].

- The Fermat's Last Theorem.

The conjecture was first proposed by Fermat in 1637. The conjecture was proven right by Andrew Wiles' in 1995 [6].

- The Goldbach's conjecture,

The conjecture states that all even numbers greater than 2 can be expressed as the sum of two primes [7-8]. İt was mention firstby Cristion Goldbach in a letter to Euler in 1742.

- The Collatz Conjecture.

The conjecture known by different names, such as the $3n + 1$ problem, the Syracuse problem, Hasse’s algorithm, etc., was first mention by Lothar Collatz in 1937.
İt states that for any positive integer: if the number is even, divide it by two and, if the number is odd, triple it and add one then the process will eventually reach the number 1, regardless of which positive integer is chosen initially [9].
İn modular arithmetic the function is defined as:

$$f(n) = \begin{cases} \frac{n}{2}, if\ n \equiv 0(mod2) \\ 3n + 1, if\ n \equiv 1(mod2) \end{cases}$$

- Twin primes conjecture.

İt states that twin primes of form $(p, p + 2)$ are infinite [10]. The term "twin prime" was coined by Paul Stäckel (1862-1919. First twin primes are (3,5), (5,7), (11,13), ...,(41,43),...

Three unsolved problems of Ancient Greece.
(600 BC - 400 AD)

Greek mathematics laid the groundwork for much of modern mathematical thought, but, considering the limitations of the knowledge and methods at the time, there were a few unresolved problems [11].

- Doubling the Cube:

The challenge was to construct a cube with a volume precisely twice that of a given cube using only a compass and straightedge. This problem dates back to the time of the Delian Oracle, where the inhabitants of Delos sought to double the size of a cube-shaped altar. The Greeks attempted to solve this geometrically, but it was proven impossible by later mathematicians using algebraic arguments. The root of the issue lies in the cube root of 2 being irrational.

- Squaring the Circle:

This problem involved constructing a square with the same area as a given circle, again only using a compass and straightedge. The ancient Greeks aimed to find a method to construct a square with an area equal to that of a given circle using a finite number of steps. This problem remained unsolved for centuries until it was proven impossible to solve using only the basic tools of geometry due to the transcendental nature of π (pi).

- Trisecting an Angle:

The challenge was to divide any angle into three equal parts using only a compass and straightedge. This problem, like the previous two, aimed at geometric constructions using specific tools. The Greeks attempted to find a way to trisect angles, but it was later proven impossible to achieve using only compass and straightedge constructions. This impossibility is linked to the algebraic nature of some trisection problems.

II. CONJECTURES &THEOREMS

Conjectures are essentially proposed ideas or statements in mathematics that are based on observations, patterns, or experimental evidence [12]. Mathematicians formulate these conjectures by examining specific cases and identifying regularities or patterns. However, to establish them as accepted truths, these conjectures need to undergo rigorous validation through mathematical proofs. When a conjecture is proven using logical reasoning, deduction, and rigorous mathematical proofs, it transforms into a theorem [13-14].

The process of proving a conjecture involves providing logical arguments and demonstrating its validity for all cases within a particular mathematical system. This process often requires creativity, deep understanding, and the application of various mathematical techniques [15-16].

Once a conjecture has been rigorously proven and accepted within the mathematical community, it becomes a theorem, adding to the body of established mathematical knowledge [13-14].

These theorems serve as fundamental building blocks in mathematics, providing the basis for further exploration, the development of new theories, and the solving of complex problems [15].

Math conjectures and proofs serve as foundational components in the education of university-level mathematics students. Conjectures, which are statements that are believed to be true but have not been proven yet, often motivate mathematical investigations [16-17]. Students explore these conjectures, attempting to find counterexamples or prove their validity [18]. This process helps students develop critical thinking, problem-solving, and reasoning skills.

When a conjecture is proven, the proof itself becomes a model for understanding the underlying principles and structures within mathematics [19-20]. These proofs demonstrate logical reasoning, the application of mathematical concepts, and the construction of rigorous arguments, all of which are essential skills for aspiring mathematicians [21-23].

Many advanced mathematical concepts and theories are built upon established conjectures and their proofs [24-25]. Studying these proofs serves as a gateway to understanding more complex mathematical ideas, providing a solid framework for students to build upon as they progress in their mathematical studies [26].

III. THE INFİNİTUDE OF THE PRİMES

This could be a great conjecture, but it was Euclid the first who gave a formal proof (known as the first ever mathematical proof) in Book IX of “The Elements”, 300 BC. At the time, symbolic algebra was not known, so all the arithmetic was cast in geometric terms [27]. At the time of Euclid, the numbers were represented as line segments, and thus only positive numbers were dealt with since operating with negative or zero lengths would be absurd then [28].

From the book “The Elements”, we have:

Definition 1. A prime number is one that is measured by a unit alone.

Definition 2. Numbers prime to one another are those that are measured by a unit alone as a common measure.

In today’s modern math, we can redefine the terms prime and relatively prime as follows:

Definition 1. An integer $p>1$ is prime if and only if the only positive divisors of p are 1 and itself.

Definition 2. Two integers a, b with at least one nonzero are relatively prime, or coprime, if they share no common factors other than 1; that is, the greatest common divisor of a and b is 1, $(a,b)=1$.

Definition 3. All natural numbers greater than 1 that are not prime as composite numbers.

Claim. Prime numbers are more than any assigned multitude of prime numbers. **Proof (Euclid).** Let A, B, and C be the assigned prime numbers. I say that there are more prime numbers than A, B, and C. Take the least number DE measured by A, B, and C. Add the unit DF to DE. Then EF is either prime or not. First, let it be prime. Then the prime numbers A, B, C, and EF have been found which are more than A, B, and C. Next, let EF not be prime. Therefore, it is measured by some prime number. Let it be measured by the prime number G. I say that G is not the same with any of the numbers A, B, and C. If possible, let it be so. Now A, B, and C measure DE, therefore G also measures DE. But it also measures EF. Therefore G, being a number, measures the remainder, the unit DF, which is absurd. Therefore, G is not the same with any one of the numbers A, B, and C. And by hypothesis it is prime. Therefore, the prime numbers A, B, C, and G have been found which are more than the assigned multitude of A, B, and C. Therefore, prime numbers are more than any assigned multitude of prime numbers.

Using today’s symbols and arguments, the proof would be:

Claim. There are infinitely many primes.

Proof. (Starting with a Lemma and several definitions and other theorems):

Lemma. Every integer greater than 1 has a prime factor (known as the Fundamental Theorem of Arithmetic”.)

Proof. The strong induction method is used.

The strong induction method is used.

For $n=2$, it is true; number 2 is prime and a factor of itself.

For other positive integers (3,4,5), the case is also true.
Assume that all the positive integers greater than 2 and less than n have a prime factor (true for positive integers greater than 1 and smaller than n).
Let's prove the case for positive integers greater than n.
We discuss two cases:
Case 1: n is prime.
Since n is a factor of itself, then it has a prime factor.
Case 2: n is not prime.
Since n is not prime, then (by the Lemma) n has a factorization of prime factors. Let's $n = a.b$, where a, (or b is prime), and both are smaller than $n, 1 < a, b < n$.
Since $a, (a < n)$, then a has a prime factor, say p.
Since p is factor of a and a is factor of n then p is factor of n; $(p/a \text{ and } a/n, \text{then } p/ n)$ and thus p is prime factor of n.

In its complete form, the Fundamental Theorem of Arithmetic is:
Theorem. Each natural number $n > 1$ can be written uniquely in the form $n = {p_1}^{a1}.{p_2}^{a2} \dots {p_r}^{ak}$, where k is a positive integer, a_i is positive integer and numbers $p_1 < p_2 < \cdots < p_k$ are distinct primes.
(Informally, the Fundamental Theorem of Arithmetic simply states that every integer greater than 1 can be written uniquely as a product of primes.)
A few other theorems are needed to continue with primes.
Theorem 1. If p is a prime number and m, n are integers such that p is factor of the product $m.n$ then p is factor of at least one of them.
If p is prime and $m, n \in Z$, and $p/ (m.n) \rightarrow p/m \text{ or } p/n$).
Theorem 2. If a prime number p is a factor of two integers m, n then p is a factor of every linear combination of m, n.
If $p/m, n \rightarrow p/(am + bn), a, b \in Z$.
Theorem 3. Any consecutive pairs of integers are relatively prime; that is, $gcd(n, n + 1) = 1$, for each integer n.

The theorem in today's notations and symbols is:
Theorem. There are infinite number of primes.
Proof.
Suppose that there are only a finite number of primes. Let the list of all primes be $\{p_1, p_2, \dots p_k\}$, from the smallest to the greatest, $(p_1 < p_2 < \cdots < p_k;\ p_1 = 2)$.
Let's consider a new number which is equal the product of all the primes plus 1:
$N = p_1.p_2 \dots p_k + 1$.
The new number N is greater than all the primes in the list could be prime or composite.
Case 1. İf the new number N is prime, then we have another prime to add to the finite list. The process repeats infinitely, and each time we have a new prime we have e larger list of primes.
Case 2. The new numer N is composite. Then, by the Fundamental Theorem of Arithmetic, the new number N has a prime factor p, which is in the finite list we already have $p \in \{p_1, p_2, \dots, p_k\}$.
Let it be $p = p_s \geq 2$, and p is a factor of N.
Since p is a factor of N and also a factor of the product $p_1.p_2 \dots p_{k,}$ then p is factor of any linear combination of them (Theorem 2), meaning p is also factor of the number $N - p_1.p_2 \dots p_k = 1$, p is a factor of 1.
As a result $p = \pm 1$ which is a contradiction, because $p \geq 2$.

Euclid's number conjecture

Definition. The product of the first n prime numbers plus 1 is called Euclid's number, $(E_n = p_1.p_2 \dots p_k + 1)$.
The first five of Euclid's numbers are also prime also. As a result we may create a conjecture:
Claim. All Euclid's number are prime.
For k=1,2,3,4,5, the claim is true.
$E_1 = 2 + 1 = 3$, prime
$E_2 = 2.3 + 1 = 7$, prime
$E_3 = 2.3.5 + 1 = 31$, prime
$E_4 = 2.3.5.7 = 211$, prime
$5 = 2.3.5.7.11 + 1 = 2311$, prime
But, the sixth Euclid number and several other are composite.
E6= 30031 composite,
E7=510511, coposite,
E8= 9699691, composite,
E9=223092871, composite,
E10= 6469693231, composite,
The conjecture remains open, being neither proved nor disproved. Besides the theoretical attempt to prove the conjecture, computer programming is

used to calculate large numbers in case patterns, formulas, or a counterexample [29-31].

Proved conjecture- turned in theorem

Fermat's last theorem was proposed by Pierre de Fermat in 1637, in the form of a note scribbled in the margin of the ancient book "Arithmetica" by Diophantus. In the note, Fermat claimed to have discovered a proof.

The conjecture states that no three positive integers $a,\ b,\ c$ satisfy the equation $a^n + b^n = c^n$ for any integer $n > 2$.

The conjecture was proved right in 1994 by Andrew Wiles and formally published in 1995 [32].

Rejected conjecture by counterexamples

In an attempt to generalize the Fermat last theorem, Leonhard Euler proposed his conjecture in 1769 [33]. It states that for all integers n and k greater than 1, if the sum of n many kth powers of positive integers is itself a kth power, then n is greater than or equal to k.

If $a_1^k + a_2^k + \cdots a_n^k = b^k \rightarrow n \geq k$.

For $n = 2$ we have the Fermat conjecture,

$a^n + b^n = c^n$ is true for $n \leq 2$.

For $n =\ 3$ the Euler conjecture is:

$a^k + b^k + c^k = d^k \rightarrow k \leq 3$.

For $n = 4$ the conjecture is:

$a^k + b^k + c^k + d^k = e^k \rightarrow k \leq 4$.

The conjecture was proved wrong by Lander and Parkin in 1966. They one counterexamples for $n = 4,\ k = 5$, using the fastest computer at the time [34].

$27^5 + 84^5 + 110^5 + 133^5 = 144^5$

The Fermat numbers

The Fermat positive numbers are of form $F_n = 2^{2^n} + 1$.

The first four numbers are F0 = 3, F1 = 5, F2 = 17, F3 = 257, F4 = 65527.

Fermat conjectured that all Fermat numbers are prime, and in 1844, another conjecture was added that said that there are infinitely many prime Fermat numbers [35].

The conjecture was rejected 100 years later, in 1732, by Euler, who calculated that the fifth number was composite.

$F5 = 2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641.6700417$

İn fact, all the known Fermat numbers for $n \geq 5$ are composite.

IV. OPEN CONJECTURES

Twin prime conjecture

The twin prime conjecture is one of the most famous conjectures that has not been proved or disproved yet [36].

Definition. Twin primes are pairs of primes of the form $(p, p + 2)$.

The first few twin primes are (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), ...

The term "twin prime" was coined by Paul Stackel in the late nineteenth century. Since that time, mathematicians have been interested in the properties of twin primes.

One of first results of looking at twin primes was the discovery that, aside from (3, 5), all twin primes are of the form $p = 6n\ \pm\ 1$.

Theorem. Twin primes $(p, p + 2),\ p \geq 5$ are of form $6k \pm 1$.

Proof. Proof. Any positive integer n, if divided by 6, is expressed as:

$n =\ 6k\ +\ r, r\ =\ 0,1,2,3,4,5,\ \ k \in N$.

In the case of prime numbers, we have only $p = 6 * k + 1{,}5$ because we have removed the values of $r = 2{,}3{,}4$.

Numbers $n = 6k + 2, 6k + 4$ are multiples of 2, and $n = 6k + 3$ is multiple of 3, so they are composite.

Because $p, p + 2$ are both prime, what is left is $p = 6k + 5$ and $p + 2\ =\ 6k + 1$, or $(p, p + 2) = 6k \pm 1$.

While there are many attempts to prove this conjecture, some significant progress has been made focusing on the gap between twin primes N [37-39].

Sophie Germain primes

Definition. A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime.

Conjecture. There are infinitely many Sophie Germain primes.

The problem is similar to the twin conjecture, and, again, with the help of computer programming, the calculation of larger numbers continues [40-41].

The first few Sophie Germain primes are 2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113,...

The conjecture, is still not proved or disproved.

Mersenne numbers

A Mersenne number is a positive integer of the form $M_n = 2^n - 1$. A Mersenne prime is a Mersenne number that is prime.

Conjecture. There exist an infinite number of Mersenne primes.

The four first Mersenne numbers, 3, 7, 31, and 127, which are all prime, were known since ancient times. The search for new Mersenne numbers restarted in the 15th century. Today, there are 51 known Mersenne numbers; some are not yet verified to be prime or composite.

Since the year 1996, all the Mersenne numbers have been discovered by GIMPS (Great Internet Mersenne Prime Search), which is a distributed computing project aimed at discovering new large prime numbers, specifically Mersenne primes [42].

The Collatz conjecture

The Collatz conjecture seems the easiest of the conjectures [43].

The Collatz Conjecture, also known as the 3n+1 problem, is a famous unsolved mathematical problem that involves iterating a particular sequence based on a simple set of rules. German mathematician Lothar Collatz first introduced the conjecture in 1937.

Collatz initially proposed the problem as a simple arithmetic game but soon realized its complexity.

Conjecture. For any given natural number, if it is even, divide by 2, and if it is odd, multiply by 3 and add 1. If we repeat the process for a finite number of iterations, the result is always 4-2-1.

The problem is known for its deceptively simple rules yet incredible complexity, which has made it challenging to prove or disprove.

The Collatz Conjecture serves as an excellent example of a deceptively simple mathematical problem that has resisted proof despite extensive exploration and computational analysis.

V. CONCLUSİON

The history of conjectures is an important part of the history of math. Such history holds significant value in mathematics and should serve as an educational and attractive topic within school curricula. All the attempts to prove or disprove conjectures have enriched the history of mathematics and often provided fresh insights into the subject. Exploring historical conjectures and persistent attempts to prove or disprove them can aid students in understanding more profoundly that a conjecture's validity is more than a limited set of observations, despite numerous examples supporting it. Exploring these conjectures allows students to understand the evolution of mathematical thought, showcasing how ideas were formulated, tested, and sometimes revised over time. Studying historical conjectures offers insights into the methods, challenges, and triumphs of mathematicians, fostering a deeper appreciation of the subject for the students while encouraging critical thinking and problem-solving skills. Working with conjectures in programming languages can be a fascinating intersection of mathematics and computational thinking. Students can use programming to explore, test, and sometimes even attempt to prove mathematical conjectures. Students can write code to simulate mathematical scenarios, generating data to test conjectures, such as testing prime number conjectures by writing algorithms to check for patterns or properties among primes. Students can use coding to assist in formalizing and verifying mathematical proofs, which involves encoding mathematical arguments into a formal language that computers can understand and checking the logical steps for accuracy. Students can develop algorithms to explore mathematical structures and look for patterns that might support or challenge conjectures. Students can develop programs or applications that visually demonstrate mathematical conjectures, which can aid in understanding. Combining programming skills with mathematical conjectures not only helps students and their professors verify or disprove mathematical ideas but also deepens understanding and opens up new avenues for research and exploration within both fields.

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