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STUDYING THE EFFICIENCY OF ROOT FINDING METHODS

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Abstract – Root finding is one of the most significant problems not only of applied mathematics, but also of engineering sciences, physics, finance etc. The implementation of efficient numerical methods to build-in functions in different software programs is a task we want to achieve. We possess different groups of methods with sufficiently good convergence order, but as we know the higher the speed is a larger amount of function and derivative evaluations per iteration is needed. The main goal in this paper is the construction of new methods with higher computational efficiency. The comparison will be made by defining the computational efficiency based on the speed of convergence, cost of evaluating the function and its derivatives and the cost of constructing the iterative process. The calculations are made using the symbolic programming language of MATLAB environment.

Keywords – Iterative Method, Order of Convergence, Efficiency Index, Root Finding.

1. Introduction

The solution of transcendent equation is one of the most investigated topics in mathematics, and because of the missing general exact solvers, the numerical methods lead the top solvers for this class of equations. Related with this aim we can use a vast literature as for example Ostrowski [1], Traub [2], Ortega and Rheinboldt [3], Neta [4], McNamee [5], Osada [6] etc. These methods can be divided into one-point and multipoint schemes. A lot of one-point iterative methods, members of Traub-Schroder sequence, which depends on f and its first *r-1* derivatives, cannot obtain an order higher than r. One of the indicators of the measuring the efficiency is the informational efficiency of one-point methods, expressed as the ratio of the order of convergence and the number of required function evaluations per iteration, cannot exceed 1. Multipoint methods give a great improvement concerning the convergence order and information and computation efficiency. During the last ten years have been published at least 200 multipoint methods [7]. Many of them turned out to be either inefficient or slight modifications/variations of already known methods. In a lot of other cases 'new' methods were only rediscovered methods. For this reason in this paper we are focused in making a systematic review of multipoint methods, concerning mainly on the most efficient methods.

2. The classification of iterative root finding methods

Let f be a real function of a real variable. If $f(\alpha)=0$ then α is a root of the equation $f(x)=0$. The following classification of iterative methods is made based on the definitions presented by Traub in [2]. We will assume that f has a certain number of continuous derivatives in the neighbourhood of the root α.

(a) Let a iterative method be of the form

$$x_{i+1}=\phi(x_i) \quad (k=0,1,2,\cdots), \tag{1}$$

where x_i is an approximation of the root α and ϕ is an iterative function. The iterative method starts with an initial approximation x_0 and at every step we use only the last known approximate. For this reason this is called one-point method.

(b) Next let x_{i+1} be determined by new information at x_i and

reused information at $x_{i-1},\cdots,x_{i-n}$. Thus

$$x_{i+1}=\phi(x_i;x_{i-1},\cdots,x_{i-n}). \tag{2}$$

Then ϕ is called one-point iterative function with memory. The semicolon in (2) separates the points at which new data are used from the points at which old data are reused. One example of iterative function with memory is the well known secant method

$$x_{i+1}=x_i-\frac{x_i-x_{i-1}}{f(x_i)-f(x_{i-1})}f(x_i),\ (i=1,2,\cdots) \tag{3}$$

(c) Another type of iteration functions is defined by using the

expression $\omega_1(x_i),\omega_2(x_i),\cdots,\omega_n(x_i)$, where x_i is the common argument. The iterative function ϕ, defined as

$$x_{i+1}=\phi(x_i,\omega_1(x_i),\omega_2(x_i),\cdots,\omega_n(x_i)), \tag{4}$$

is called a multipoint iterative function without memory. The simplest examples are Steffensen's method

$$x_{i+1}=x_i-\frac{f^2(x_i)}{f(x_i+f(x_i))-f(x_i)} \quad \text{with}$$

$$\omega_1(x_i)=x_i+f(x_i) \tag{5}$$

and Traub-Steffensen's method

$$x_{i+1} = x_i - \frac{\gamma f^2(x_i)}{f(x_i + \gamma f(x_i)) - f(x_i)} \text{ with}$$
$$\omega_1(x_i) = x_i + \gamma f(x_i) \quad (6)$$

(d) Finally, if x_{i+1} is determined by new information at $x_{i-1}, \cdots, x_{i-k}$

and reused information $x_{i-k-1}, \cdots, x_{i-n}$ the iterative function

$$x_{i+1} = \phi(x_i, x_{i-1}, \cdots, x_{i-k}; x_{i-k-1}, \cdots, x_{i-n}),\ n > k, \quad (7)$$

is called a multipoint iterative function with memory. The semicolon in (7) separates the points at which new data are used from the points at which old data are reused. There are no well-known methods of multipoint iterative functions with memory. In this paper we will treat some new improvements of multipoint methods without and with memory for finding a simple zero.

3. Multipoint iterative methods

One of the most important features to determine the advantages of on iterative method is the convergence rate determined by the order of convergence. Let $\{x_i\}$ be a sequence that converges to α and let $\varepsilon_i = x_i - \alpha$. If there exists a real number *p* and a nonzero positive constant C_p such that

$$\lim_{i\to\infty} \frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} = C_p,$$

then *p* is called the order of the sequence $\{x_i\}$ and C_p is the asymptotic error constant. In practise, when testing new methods is of interest to use the computational order of convergence defined by

$$\tilde{r} = \frac{\log|(x_i - \alpha)/(x_{i-1} - \alpha)|}{\log|(x_{i-1} - \alpha)/(x_{i-2} - \alpha)|}, \quad (8)$$

where x_{i-2}, x_{i-1} and x_i are the last three successive approximations to the root α. Also this formula is only of theoretical value, since the value of the zero α is unknown in practice. Using the factorization $f(x) = (x-\alpha)g(x)$ and (8), is derived the approximate formula for computational order of convergence

$$r_c = \frac{\log|f(x_i)/f(x_{i-1})|}{\log|f(x_{i-1})/f(x_{i-2})|}, \quad (9)$$

which is of much better practical importance.

There are other measures for comparing various iterative techniques. Traub [2] introduces the informational efficiency and efficiency index, which can be expressed in terms of the order (r) of the method and the number of function-(and derivative) evaluations (θ_f). The informational efficiency of an iterative method (M) is defined as

$$I(M) = \frac{r}{\theta_f}. \quad (10)$$

The efficiency index (or computational efficiency) is given by

$$E(M) = r^{1/\theta_f}, \quad (11)$$

the definition that was introduces by Ostrowski [1].

One major goal in designing new numerical methods is to obtain a method with the best possible computational efficiency. Each memory-free iteration consists of

- New function evaluation, and
- Arithmetic operations used to combine the available data.

Minimizing the total number of arithmetic operations through an iterative process which would provide the zero approximation of the desired accuracy, would be very much dependent on the particular properties of a function f whose zero is sought. Regarding the definition (10) and (11), this means that it is desirable to achieve as high as possible convergence order with the fixed number of function evaluations per iteration.

Kung and Traub (1974) stated the following derivative free family of iterative methods without memory.

For an initial approximation x_0, arbitrary $n \in N$ and $k = 0,1,\cdots$ define the iteration function $\psi_j(f)$ $(j = -1,0,1,\cdots)$ as follows:

$$\begin{cases} y_{k,0} = \psi_0(f)(x_k) = x_k,\ y_{k,-1} = \psi_{-1}(f)(x_k) = x_k + \gamma_k f(x_k) \\ y_{k,j} = \psi_j(f)(x_k) = R_j(0),\ j = 1,\cdots,n \\ x_{k+1} = y_{k,n} = \psi_n(f)(x_k) \end{cases} \quad (12)$$

where $R_j(\tau)$ represents an inverse interpolatory polynomial of degree no greater then *j*.

Zheng et al. proposed in [7] other derivative free family of n-point methods of arbitrary order of convergence $\gamma_k \in R^* \ 2^n \ (n \ge 1)$. This family is constructed using Newton's interpolation with forward divided differences.

For an initial approximation x_0, arbitrary $n \in N$, $\gamma_k \in R^*$ and $k = 0,1,\cdots$ the *n*-th point method is defined by

$$y_{k,0}, y_{k-1,j_1}, \cdots, y_{k-1,j_m}$$

$$y_{k,1} = y_{k,0} - \frac{f(y_{k,0})}{f[y_{k,0}, y_{k-1}]} \quad (13)$$

$$y_{k,2} = y_{k,1} - \frac{f(y_{k,1})}{f[y_{k,1}, y_{k,0}] + f[y_{k,1}, y_{k,0}, y_{k-1}](y_{k-1} - y_{k,0})}$$

$$y_{k,n} = y_{k,n-1} - \frac{f(y_{k,n-1})}{f[y_{k,n-1}, y_{k,n-2}] + \sum_{j=1}^{n-1} f[y_{k,n-1}, \cdots, y_{k,n-2-j}] \prod_{i=1}^{j} (y_{k,n-1} - y_{k,n-1-i})}$$

$$x_{k+1} = y_{k,n}.$$

The order of the convergence of the families (12) and (13) is 2^n. Since these families require $n+1$ function evaluations, they are optimal.

Now we will show that both families can be extremely accelerated without any addition function evaluations. The construction of the new families of n-point derivative free methods is based on the variation of a free parameter γ_k in each iterative step. This parameter is calculated using information from the current and previous iteration so that the presented methods may be regarded as methods with memory. It is evident that the order of these families would be $2^n + 2^{n-1}$ if we could provide $\gamma_k = -1/f'(\alpha)$. The value $f'(\alpha)$ is not known in practice and we will substitute it with an approximation, calculated using available information. We present the following model for approximating $f'(\alpha)$:

$\tilde{f}'(\alpha) = N'_m(y_{k,0})$ (Newton's interpolation with divided differences),

where

$$N_m(\tau) = N_m(\tau; y_{k,0}, y_{k-1,j_1}, \cdots, y_{k-1,j_m}), -1 \le j_m < j_{m-1} < \cdots < j_1 \le n-1 \quad (14)$$

represents Newton's interpolating polynomial of degree m $(1 \le m \le n-1)$, set through m+1 available approximations (nodes) $y_{k,0}, y_{k-1,j_1}, \cdots, y_{k-1,j_m}$.

From Table 1 we observe that the order of convergence of the families (12) and (13) with memory is considerably increased relative to the corresponding basic families without memory (entries in the last row). The increment in percentage is also displayed and we can see that the improvement of the order is up to 50%. It is worth noting that the improvement of the convergence order is obtained without addition function evaluations.

Table 1. The lower bound of the convergence order given in bold.

n	1	2	3	4
m=1				
j=0	**2.414** (20.7%)	**4.449**(11.2%)	**8.472**(6%)	**16.485**(3%)
j=1		**5**(25%)	**9**(12.5%)	**17**(6.25%)
j=2			**10**(25%)	**18**(12.5%)
j=3				**20**(25%)
m=2	**3**(50%)	**5.372**(34%)	**11**(37.5%)	**22**(37.5%)
m=3		**6**(50%)	**11.35**(41.9%)	**23**(43.7%)
without memory	**2**	**4**	**8**	**16**

Conclusions

In this paper we presented the advantages of multipoint iterative methods for solving nonlinear equations. We presented two well-known families of multipoint methods with memory and gave some new improvements in these families and more preciously in the approximation of a parameter, for which we use the Newton divided differences, for obtaining a higher order of convergence without using any new function evaluations.

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