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Research Article

Some inequalities on functional Hilbert space operators

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Abstract – Several Berezin radius and norm inequalities for functional Hilbert space operators are provided in this study. Some previous comparable inequalities are improved by these inequalities. We show that

$$\mathfrak{ber}^2(\mathfrak{A}_1) \le \frac{1}{2}\left\| |\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4 + \frac{1}{2}\left(|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2\right)^2 \right\|_{\mathfrak{ber}}^{1/2}$$

for an operator $\mathfrak{A}_1$.

Keywords – Functional Hilbert Space, Berezin Symbol, Positive Operator, Berezin Radius, Inequality.

I. INTRODUCTION

The definition of its Berezin symbol $\widetilde{\mathfrak{A}_1}$ for every bounded linear operator $\mathfrak{A}_1$ on $\mathfrak{H}$ (i.e., $\mathfrak{A}_1 \in \mathfrak{B}(\mathfrak{H})$, the Banach algebra of every bounded linear operators on $\mathfrak{H}$, is (see Berezin [1])

$$\widetilde{\mathfrak{A}_1}(\iota) := \langle \mathfrak{A}_1 \hat{\varkappa}_\iota, \hat{\varkappa}_\iota \rangle,\ \iota \in \digamma.$$

where $\hat{\varkappa}_\iota(z) := \frac{\varkappa_\iota(z)}{\|\varkappa_\iota\|_{\mathcal{H}}}$ is the normalized reproducing kernel of $\mathfrak{H}$.

The set

$$\mathfrak{Ber}(\mathfrak{A}_1) = \mathcal{Range}\left(\widetilde{\mathfrak{A}_1}\right) = \left\{\widetilde{\mathfrak{A}_1}(\iota) : \iota \in \digamma\right\}$$

is termed as Berezin set of operator $\mathfrak{A}_1$ and the definition of its Berezin radius, $\mathfrak{ber}(\mathfrak{A}_1)$, is

$$\mathfrak{ber}(\mathfrak{A}_1) = \sup_{\iota \in \digamma}\left|\widetilde{\mathfrak{A}_1}(\iota)\right|.$$

(see Karaev [7]).

The Berezin set $\mathfrak{Ber}(\mathfrak{A}_1)$ is a subset of the numerical range $W(\mathfrak{A}_1) = \{\langle \mathfrak{A}_1 x, x\rangle : x \in \mathfrak{H} \text{ and } \|x\| = 1\}$ and it is clear that

$$\mathfrak{ber}(\mathfrak{A}_1) \le w(\mathfrak{A}_1) := \sup_{x\in\mathfrak{H}, \|x\|=1} |\langle \mathfrak{A}_1 x, x\rangle|,$$

which is $\mathfrak{A}_1$'s numerical radius. Therefore, employing these new qualities to examine new properties of operator $\mathfrak{A}_1$ will be natural and intriguing. See [5], [8], and [10] for several recent and striking conclusions using inequalities for the numerical radius.

An essential observation to make is that:

$$\mathfrak{ber}(\mathfrak{A}_1) \le w(\mathfrak{A}_1) \le \|\mathfrak{A}_1\|, \tag{1.1}$$

and

$$\mathfrak{ber}(\mathfrak{A}_1) \le \|\mathfrak{A}_1\|_{\mathfrak{ber}} \le \|\mathfrak{A}_1\|, \forall \mathfrak{A}_1 \in \mathfrak{B}(\mathfrak{H}). \tag{1.2}$$

Huban et al. [6] refine the inequality (1.2) by developing the following inequality

$$\mathfrak{ber}^2(\mathfrak{A}_1) \le \frac{1}{2}\left\| |\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2 \right\|_{\mathfrak{ber}}. \tag{1.3}$$

where $|\mathfrak{A}_1| = (\mathfrak{A}_1^*\mathfrak{A}_1)^{1/2}$. The second disparity in (1.1) is less severe than the inequality in (1.2).

Section 2 considerably improves the second inequality in (1.2). We also give a multiplicative refinement of this inequality.

In order to achieve the purpose of this paper, we require the following results.

If $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{B}(\mathfrak{H})$ are positive operators, then

$$\|\mathfrak{A}_1 + \mathfrak{A}_2\| \le \frac{1}{2}\left(\|\mathfrak{A}_1\| + \|\mathfrak{A}_2\| + \sqrt{(\|\mathfrak{A}_1\| - \|\mathfrak{A}_2\|)^2 + 4\left\| \mathfrak{A}_1^{\frac{1}{2}} - \mathfrak{A}_2^{\frac{1}{2}} \right\|^2} \right) \tag{1.4}$$

(see [8]).

Buzano's inequality (see [3]): If a, b, x are vectors in $\mathfrak{H}$, then

$$|\langle a, x\rangle||\langle x, b\rangle| \le \frac{\|a\|\|b\| + |\langle a, b\rangle|}{2}\|x\|^2. \tag{1.5}$$

For $\mathfrak{A}_1 \in \mathfrak{B}(\mathfrak{H})$, $x, y \in \mathbb{H}$ vectors and $t \in [0,1]$, we have

$$|\langle \mathfrak{A}_1 x, y\rangle|^2 \le \left\langle |\mathfrak{A}_1|^{2(1-t)} x, x \right\rangle \le \langle |\mathfrak{A}_1^*| y, y, \rangle \tag{1.6}$$

(see [9]).

The Cauchy-Schwarz inequality gives us

$$\langle \mathfrak{A}_1 x, x\rangle^2 \le \langle \mathfrak{A}_1^2 x, x\rangle \tag{1.7}$$

for $\mathfrak{A}_1 \in \mathfrak{B}(\mathfrak{H})$ self-adjoint operator and $x \in \mathfrak{H}$ unit vector.

II. MAIN RESULTS

Our Berezin radius inequality, which includes many inequalities as special instances, is proven.

Theorem 1. *If* $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{B}(\mathfrak{H})$, *then*

$$\mathrm{ber}(\mathfrak{A}_1 \pm \mathfrak{A}_2) \leq \frac{1}{2}\sqrt{\||3(|\mathfrak{A}_1|^2 + |\mathfrak{A}_2|^2) + |\mathfrak{A}_1^*|^2 + |\mathfrak{A}_2^*|^2\| + 2\left(\mathrm{ber}(\mathfrak{A}_1^2) + \mathrm{ber}(\mathfrak{A}_2^2) + 2\mathrm{ber}(\mathfrak{A}_2^*\mathfrak{A}_1)\right)}$$

and

$$\mathrm{ber}(\mathfrak{A}_1 \pm \mathfrak{A}_2) \leq \frac{1}{2}\sqrt{\|3(|\mathfrak{A}_1^*|^2 + |\mathfrak{A}_2^*|^2) + |\mathfrak{A}_1|^2 + |\mathfrak{A}_2|^2\| + 2\left(\mathrm{ber}(\mathfrak{A}_1^2) + \mathrm{ber}(\mathfrak{A}_2^2) + 2\mathrm{ber}(\mathfrak{A}_2\mathfrak{A}_1^*)\right)}$$

More precisely,

$$\mathrm{ber}^2(\mathfrak{A}_1 \pm \mathfrak{A}_2) \leq \frac{1}{2}\left(\mathrm{ber}(A_1^2) + \mathrm{ber}(\mathfrak{A}_2^2)\right) + \min\{\varsigma, \tau\}$$

where $\varsigma = \frac{1}{4}\|3(|\mathfrak{A}_1|^2 + |\mathfrak{A}_2|^2) + |\mathfrak{A}_1^*|^2 + |\mathfrak{A}_2^*|^2\|_{\mathrm{ber}} + \mathrm{ber}(\mathfrak{A}_2^*\mathfrak{A}_1)$ *and* $\tau = \frac{1}{4}\|3(|\mathfrak{A}_1^*|^2 + |\mathfrak{A}_2^*|^2) + |\mathfrak{A}_1|^2 + |\mathfrak{A}_2|^2\|_{\mathrm{ber}} + \mathrm{ber}(\mathfrak{A}_2\mathfrak{A}_1^*)$.

Proof. By taking $a = \mathfrak{A}_1\hat{\kappa}_\iota$, $b = \mathfrak{A}_2\hat{\kappa}_\iota$ with $\iota \in \digamma$ in (1.5) and the AM-GM inequality, then we get

$$\begin{aligned}
&2||\langle \mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle||\langle \hat{\kappa}_\iota, \mathfrak{A}_2\hat{\kappa}_\iota\rangle| \\
&= 2|\langle \mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle||\langle \mathfrak{A}_2\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle| \\
&\leq \|\mathfrak{A}_1\hat{\kappa}_\iota\|_{\mathrm{ber}}\|\mathfrak{A}_2\hat{\kappa}_\iota\|_v + |\langle \mathfrak{A}_1\hat{\kappa}_\iota, A_2\hat{\kappa}_\iota\rangle| \\
&= \|\mathfrak{A}_1\hat{\kappa}_\iota\|_{\mathrm{ber}}\|\mathfrak{A}_2\hat{\kappa}_\iota\|_{\mathrm{ber}} + |\langle \mathfrak{A}_2^*\mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle| \\
&= \sqrt{\langle \mathfrak{A}_1\hat{\kappa}_\iota, \mathfrak{A}_1\hat{\kappa}_\iota\rangle\langle \mathfrak{A}_2\hat{\kappa}_\iota, \mathfrak{A}_2\hat{\kappa}_\iota\rangle} + |\langle \mathfrak{A}_2^*A_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle| \\
&= \sqrt{\langle |\mathfrak{A}_1|^2\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle\langle |\mathfrak{A}_2|^2\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle} + |\langle \mathfrak{A}_2^*\mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle| \\
&\leq \frac{1}{2}\left(\langle |\mathfrak{A}_1|^2\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle + \langle |\mathfrak{A}_2|^2\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle\right) + |\langle \mathfrak{A}_2^*A_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle| \\
&= \frac{1}{2}\langle (|\mathfrak{A}_1|^2 + |\mathfrak{A}_2|^2)\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle + |\langle \mathfrak{A}_2^*\mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle|.
\end{aligned}$$

Observe that

$$|\langle \mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle||\langle \mathfrak{A}_2\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle| \leq \frac{1}{4}\langle (|\mathfrak{A}_1|^2 + |\mathfrak{A}_2|^2)\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle + \frac{1}{2}|\langle \mathfrak{A}_2^*\mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle| \qquad (2.1)$$

In particular, if $\mathfrak{A}_2^* = \mathfrak{A}_1$, then

$$|\langle \mathfrak{A}_1\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle|^2 \leq \frac{1}{4}\langle (|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2)\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle + \frac{1}{2}|\langle \mathfrak{A}_1^2\hat{\kappa}_\iota, \hat{\kappa}_\iota\rangle|. \qquad (2.2)$$

From the triangle inequality, the inequalities (2.1) and (2.2), we get

$$\begin{aligned}
&|\langle(\mathfrak{A}_1+\mathfrak{A}_2)\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|^2\\
&\leq (|\langle\mathfrak{A}_1\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|+|\langle\mathfrak{A}_2\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|)^2\\
&= |\langle\mathfrak{A}_1\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|^2+|\langle\mathfrak{A}_2\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|^2+2|\langle\mathfrak{A}_1\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle||\langle\mathfrak{A}_2\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|\\
&\leq \frac{1}{2}(|\langle\mathfrak{A}_1^2\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|+|\langle\mathfrak{A}_2^2\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|+2|\langle\mathfrak{A}_2^*\mathfrak{A}_1\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|)\\
&+\frac{1}{4}\langle(3(|\mathfrak{A}_1|^2+|\mathfrak{A}_2|^2)+|\mathfrak{A}_1^*|^2+|\mathfrak{A}_2^*|^2)\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle\\
&\leq \frac{1}{2}\left(\mathfrak{ber}(\mathfrak{A}_1^2)+\mathfrak{ber}(\mathfrak{A}_2^2)+2\mathfrak{ber}(\mathfrak{A}_2^*\mathfrak{A}_1)\right)\\
&+\frac{1}{4}\|3(|\mathfrak{A}_1|^2+|\mathfrak{A}_2|^2)+|\mathfrak{A}_1^*|^2+|\mathfrak{A}_2^*|^2\|_{\mathfrak{ber}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathfrak{ber}^2(\mathfrak{A}_1+\mathfrak{A}_2) &= \sup_{\iota\in\mathfrak{f}}|\langle(\mathfrak{A}_1+\mathfrak{A}_2)\hat{\kappa}_\iota,\hat{\kappa}_\iota\rangle|^2\\
&\leq \frac{1}{2}\left(\mathfrak{ber}(\mathfrak{A}_1^2)+\mathfrak{ber}(\mathfrak{A}_2^2)+2\mathfrak{ber}(\mathfrak{A}_2^*\mathfrak{A}_1)\right)\\
&+\frac{1}{4}\|3(|\mathfrak{A}_1|^2+|\mathfrak{A}_2|^2)+|\mathfrak{A}_1^*|^2+|\mathfrak{A}_2^*|^2\|_{\mathfrak{ber}}
\end{aligned}$$

and so

$$\begin{aligned}
\mathfrak{ber}^2(\mathfrak{A}_1+\mathfrak{A}_2) &\leq \frac{1}{2}\left(\mathfrak{ber}(\mathfrak{A}_1^2)+\mathfrak{ber}(\mathfrak{A}_2^2)+2\mathfrak{ber}(\mathfrak{A}_2^*\mathfrak{A}_1)\right)\\
&+\frac{1}{4}\|3(|\mathfrak{A}_1|^2+|\mathfrak{A}_2|^2)+|\mathfrak{A}_1^*|^2+|\mathfrak{A}_2^*|^2\|_{\mathfrak{ber}}.
\end{aligned}$$

If we substitute $-\mathfrak{A}_2$ for $\mathfrak{A}_2$ in the previous inequality, we obtain the first inequality. By substituting $\mathfrak{A}_1^*$ and $\mathfrak{A}_2^*$ for $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, the first inequality may be converted into the second inequality. The first and second inequality make the third inequality clear. □

Corollary 1. *(i) In case $\mathfrak{A}_2=\mathfrak{A}_1$ in Theorem 1, we have*

$$\mathfrak{ber}^2(\mathfrak{A}_1)\leq\frac{1}{4}(\mathfrak{ber}(\mathfrak{A}_1^2)+\|\mathfrak{A}_1\|_{\mathfrak{ber}}^2)+\frac{1}{8}\|3|\mathfrak{A}_1|^2+|\mathfrak{A}_1^*|^2\|_{\mathfrak{ber}}$$

(ii) In case $\mathfrak{A}_2=\mathfrak{A}_1^*$ in Theorem 1, we have

$$\|\Re\mathfrak{A}_1\|_{\mathfrak{ber}}^2\leq\frac{1}{2}\mathfrak{ber}(\mathfrak{A}_1^2)+\frac{1}{4}\||\mathfrak{A}_1|^2+|\mathfrak{A}_1^*|^2\|_{\mathfrak{ber}},$$

where $\Re\mathfrak{A}_1=\frac{\mathfrak{A}_1+\mathfrak{A}_1^*}{2}$.

When we establish the triangle inequality using the inner product approaches, the following result is readily reached. To be more specific, the direct evidence is: Using the polar decomposition $\mathfrak{A}_1=\mathbb{U}|\mathfrak{A}_1|$, assume that $\mathfrak{A}_1\in\mathfrak{B}(\mathfrak{H})$. Next,

$$\mathfrak{ber}(\mathfrak{A}_1)=\mathfrak{ber}\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}+\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right)\leq\mathfrak{ber}\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right)+\frac{\|\mathfrak{A}_1\|}{2}\mathfrak{ber}(\mathbb{U}).$$

Theorem 2. *Let $\mathfrak{A}_1\in\mathfrak{B}(\mathfrak{H})$ with the polar decomposition $\mathfrak{A}_1=\mathbb{U}|\mathfrak{A}_1|$. Then we have*

$$\mathfrak{ber}(\mathfrak{A}_1)\leq\mathfrak{ber}\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|_{\mathfrak{ber}}}{2}\mathbb{U}\right)+\frac{\|\mathfrak{A}_1\|_{\mathfrak{ber}}}{2}\mathfrak{ber}(\mathbb{U}).$$

Proof. *[2, Th. 3.3] has demonstrated that*

$$\begin{aligned}|\langle|\mathfrak{A}_1|^2\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle| &\le \left|\langle|\mathfrak{A}_1|^2\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle-\frac{\|\mathfrak{A}_1\|^2}{2}\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle\right|+\frac{\|\mathfrak{A}_1\|^2}{2}\left|\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle\right|\\ &\le\frac{\|\mathfrak{A}_1\|^2}{2}(|\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle|+\|\hat{\varkappa}_\iota\|\|\hat{\varkappa}_\kappa\|).\end{aligned}$$

This may be expressed as follows:

$$\begin{aligned}|\langle|\mathfrak{A}_1|^2\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle| &\le \left|\langle|\mathfrak{A}_1|^2\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle-\frac{\||\mathfrak{A}_1|^2\|}{2}\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle\right|+\frac{\||\mathfrak{A}_1|^2\|}{2}\left|\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle\right|\\ &\le\frac{\||\mathfrak{A}_1|^2\|}{2}(|\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle|+\|\hat{\varkappa}_\iota\|\|\hat{\varkappa}_\kappa\|).\end{aligned}$$

Hence, by replacing $|\mathfrak{A}_1|^2$ *by* $|\mathfrak{A}_1|$, *we have*

$$\begin{aligned}|\langle|\mathfrak{A}_1|\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle| &\le \left|\langle|\mathfrak{A}_1|\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle-\frac{\|\mathfrak{A}_1\|}{2}\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle\right|+\frac{\|\mathfrak{A}_1\|}{2}\left|\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle\right|\\ &\le\frac{\|\mathfrak{A}_1\|}{2}(|\langle\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle|+\|\hat{\varkappa}_\iota\|\|\hat{\varkappa}_\kappa\|).\end{aligned}\tag{2.3}$$

If we replace $\hat{\varkappa}_\kappa$ *with* $\mathbb{U}^*\hat{\varkappa}_\kappa$, *in (2.3), we obtain*

$$\begin{aligned}|\langle\mathfrak{A}_1\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle| &\le \left|\left\langle\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right)\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\right\rangle\right|+\frac{\|\mathfrak{A}_1\|}{2}|\langle\mathbb{U}\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle|\\ &\le\frac{\|\mathfrak{A}_1\|}{2}(|\langle\mathbb{U}\hat{\varkappa}_\iota,\hat{\varkappa}_\kappa\rangle|+\|\hat{\varkappa}_\iota\|\|\mathbb{U}^*\hat{\varkappa}_\kappa\|).\end{aligned}$$

Specifically,

$$\begin{aligned}|\langle\mathfrak{A}_1\hat{\varkappa}_\iota,\hat{\varkappa}_\iota\rangle| &\le \left|\left\langle\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right)\hat{\varkappa}_\iota,\hat{\varkappa}_\iota\right\rangle\right|+\frac{\|\mathfrak{A}_1\|}{2}|\langle\mathbb{U}\hat{\varkappa}_\iota,\hat{\varkappa}_\iota\rangle|\\ &\le\mathfrak{ber}\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right)+\frac{\|\mathfrak{A}_1\|}{2}\mathfrak{ber}(\mathbb{U})\end{aligned}$$

for every $\iota\in\digamma$, *i.e.,* $|\langle\mathfrak{A}_1\hat{\varkappa}_\iota,\hat{\varkappa}_\iota\rangle|\le\mathfrak{ber}\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right)+\frac{\|\mathfrak{A}_1\|}{2}\mathfrak{ber}(\mathbb{U})$.

By taking supremum over all $\iota\in\digamma$, *we infer that*

$$\mathfrak{ber}(\mathfrak{A}_1)\le\mathfrak{ber}\left(\mathfrak{A}_1-\frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right)+\frac{\|\mathfrak{A}_1\|}{2}\mathfrak{ber}(\mathbb{U})$$

as desired. □

We can give the following lemmas without proof using simple calculations, Lemmas 2.4 and 2.5 in [5].

Lemma 1. *Let* $\mathfrak{A}_1\in\mathfrak{B}(\mathfrak{H})$. *Then*

$$\left\||\mathfrak{A}_1|-\frac{\|\mathfrak{A}_1\|}{2}\mathfrak{I}\right\|_{\mathfrak{ber}}=\frac{\|\mathfrak{A}_1\|_{\mathfrak{ber}}}{2}$$

where $\mathfrak{I}\in\mathfrak{B}(\mathfrak{H})$ *is the identity operator.*

Lemma 2. *If* $\mathfrak{A}_1\in\mathfrak{B}(\mathfrak{H})$ *with the polar decomposition* $\mathfrak{A}_1=\mathbb{U}|\mathfrak{A}_1|$. *Then*

$$\left\|\mathfrak{A}_1 - \frac{\|\mathfrak{A}_1\|}{2}\mathbb{U}\right\|_{\mathfrak{ber}} = \frac{\|\mathfrak{A}_1\|_{\mathfrak{ber}}}{2}.$$

Using inequalities (1.6) and (1.7), we provide the following new bound to wrap up this section.

Theorem 3. *For any* $\mathfrak{A}_1 \in \mathfrak{B}(\mathfrak{H})$ *and* $t \in [0,1]$, *we have that*

$$\mathfrak{ber}^2(\mathfrak{A}_1) \leq \frac{1}{2}\left\||\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4 + \frac{1}{2}(|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2)^2\right\|_{\mathfrak{ber}}^{\frac{1}{2}}.$$

Proof. We imitate a few concepts from [11,12]. For $t \in [0,1]$, we get

$$\begin{aligned}
&\left((1-t)|\mathfrak{A}_1|^2 + t|\mathfrak{A}_1^*|^2\right)^2 \\
&= \left((1-2t)|\mathfrak{A}_1|^2 + 2t\left(\frac{|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2}{2}\right)\right)^2 \\
&\leq (1-2t)|\mathfrak{A}_1|^4 + 2t\left(\frac{|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2}{2}\right)^2
\end{aligned}$$

$$= (1-t)|\mathfrak{A}_1|^4 + t|\mathfrak{A}_1^*|^4 - 2r\left(\frac{|\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4}{2} - \left(\frac{|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2}{2}\right)^2\right)$$

For $1/2 \leq t \leq 1$, the discussion is identical. Hence we deduce that

$$\begin{aligned}
&\left((1-t)|\mathfrak{A}_1|^2 + t|\mathfrak{A}_1^*|^2\right)^2 \\
&\leq (1-t)|\mathfrak{A}_1|^4 + t|\mathfrak{A}_1^*|^4 - 2r\left(\frac{|\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4}{2} - \left(\frac{|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2}{2}\right)^2\right)
\end{aligned}$$

Also, from the inequality (1.6), [4, Lemma 3] and the weighted AM-GM inequality, we deduce that

$$\begin{aligned}
|\langle \mathfrak{A}_1\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\rangle|^4 &\leq \left(\langle |\mathfrak{A}_1|^{2(1-t)}\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\rangle \left\langle |\mathfrak{A}_1^*|^{2t}\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\right\rangle\right)^2 \\
&\leq (\langle |\mathfrak{A}_1|^2\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\rangle^{1-t}\langle |\mathfrak{A}_1^*|^2\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\rangle^t)^2 \\
&\leq ((1-t)\langle |\mathfrak{A}_1|^2\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\rangle + t\langle |\mathfrak{A}_1^*|^2\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\rangle)^2
\end{aligned}$$

and from (1.7)

$$\begin{aligned}
&= \left\langle \left((1-t)|\mathfrak{A}_1|^2 + t|\mathfrak{A}_1^*|^2\right)\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\right\rangle^2 \\
&\leq \left\langle \left((1-t)|\mathfrak{A}_1|^2 + t|\mathfrak{A}_1^*|^2\right)^2\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\right\rangle.
\end{aligned}$$

So, we have

$$\begin{aligned}
&|\langle \mathfrak{A}_1\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\rangle|^4 \\
&\leq \left\langle \left((1-t)|\mathfrak{A}_1|^4 + t|\mathfrak{A}_1^*|^4 - 2r\left(\frac{|\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4}{2} - \left(\frac{|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2}{2}\right)^2\right)\right)\hat{\varkappa}_\iota, \hat{\varkappa}_\iota\right\rangle,
\end{aligned}$$

$$|\langle \mathfrak{A}_1 \hat{\varkappa}_\iota, \hat{\varkappa}_\iota \rangle|^4$$
$$\leq \left\langle \left(\frac{1}{2}(|\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4) - \frac{1}{2}\left(\frac{|\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4}{2} - \left(\frac{|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2}{2} \right)^2 \right) \right) \hat{\varkappa}_\iota, \hat{\varkappa}_\iota \right\rangle$$
$$= \frac{1}{4}\left\langle \left(|\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4 + \frac{1}{2}(|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2)^2 \right) \hat{\varkappa}_\iota, \hat{\varkappa}_\iota \right\rangle$$
$$\leq \frac{1}{4}\left\| |\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4 + \frac{1}{2}(|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2)^2 \right\|$$

and

$$|\langle \mathfrak{A}_1 \hat{\varkappa}_\iota, \hat{\varkappa}_\iota \rangle|^4 \leq \frac{1}{4}\left\| |\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4 + \frac{1}{2}(|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2)^2 \right\|.$$

By taking supremum over all $\iota \in \digamma$, we deduce that

$$\mathfrak{ber}^4(\mathfrak{A}_1) \leq \frac{1}{4}\left\| |\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4 + \frac{1}{2}(|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2)^2 \right\|_{\mathfrak{ber}}$$

or

$$\mathfrak{ber}^2(\mathfrak{A}_1) \leq \frac{1}{2}\left\| |\mathfrak{A}_1|^4 + |\mathfrak{A}_1^*|^4 + \frac{1}{2}(|\mathfrak{A}_1|^2 + |\mathfrak{A}_1^*|^2)^2 \right\|_{\mathfrak{ber}}^{\frac{1}{2}}.$$

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