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Research Article

New Analytical Wave Solutions and Physical Interpretations of the Benjamin–Ono Equation

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Abstract **–**In this study, the G'/G^2 method, one of the methods used to find analytical solutions of nonlinear partial differential equations, is analysed. With the use of this method, travelling wave solutions of the Benjamin Ono equation have been found and various solutions have been obtained depending on arbitrary parameters. In this paper, the applicability and efficiency of this method on nonlinear evolution equations (NLEEs) are investigated.

Keywords – Benjamin Ono Equation, G'/G^2 Method, Travelling Wave Solutions, Exact Solution.

I. INTRODUCTION

Nonlinear partial differential equations (PDEs) occupy a central position in the modelling of complex physical phenomena. They provide an effective means of representing dispersive, solitonic or chaotic behaviours that are often inadequately described by classical linear models. Especially in fields such as fluid dynamics, pulse propagation in optical fibres, quantum field theory and plasma physics, such equations form the fundamental building blocks of natural systems [1-7]. The Benjamin-Ono (BO) equation considered in this study is a nonlinear dispersive equation developed to model the propagation of one-dimensional, internal gravity waves in shallow water. The general form of the equation is expressed as follows:

$$u_t + Hu_{xx} + uu_x = 0. \quad (1)$$

The BO equation describes the propagation of long internal waves, especially in bilayer liquid systems, and differs from classical models such as the Korteweg-de Vries (KdV) equation due to the fractional nature of dispersion. While the KdV equation models shorter-range dispersions, the BO equation includes dispersive effects propagating throughout the entire space.

The equation was introduced independently by T. Brooke Benjamin and Hiroaki Ono to model the propagation of shallow water waves and internal waves [8, 9]. Later, the inverse scattering transform (IST) method was developed for this equation and multiple soliton solutions were obtained [10]. Bekir obtained many solutions for BO and similar equations by G′/G method [11] and obtained trigonometric and hyperbolic solutions. However, since some of these methods require quite complex technical analyses, G′/G and its derivative methods, which offer a more systematic solution path, stand out as an important alternative. G′/G and G′/G^2 methods have been successfully applied to many nonlinear partial differential equations in recent years. They were first proposed by Wang et al. [12] and then developed by researchers such as Yomba [13], Kudryashov [14]. These methods allow to obtain analytically tangent, hyperbolic, exponential or rational solutions by associating the solution search with an auxiliary quadratic linear ODE. In integrable systems such as the BO equation, these techniques have contributed to the systematic finding of soliton, periodic and rational solutions.

The main objective of this paper is to apply the modified analytical solution technique known as the G'/G^2 method to the Benjamin-Ono equation in order to systematically obtain various types of analytical solutions of the equation (e.g. soliton, rational, periodic or hyperbolic solutions). The G'/G^2 method is a derivative of the classical G′/G method for the solution of nonlinear differential equations, and it allows for a wider set of solutions.. The G'/G^2 method is a derivative of the classical G′/G method for solving nonlinear differential equations and can cover a wider set of solutions.

In this study, firstly the structure and physical interpretation of the Benjamin-Ono equation will be emphasised, then the G'/G^2 method will be introduced in detail. Then, the analytical solutions obtained by using this method will be presented and the physical meanings of the obtained results will be evaluated by comparing them with similar studies in the literature. Thus, this study aims both to demonstrate the effectiveness of the method and to expand the range of solutions for the Benjamin-Ono equation.

II. MATERIALS AND METHOD

Analysis of G'/G^2 *- Expansion Method*

Let us consider a general nonlinear partial differential equation (PDE) of the form:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \tag{2}$$

where $u = u(x, y, \dots)$ is the unknown function, and P is a polynomial in u and its derivatives.

Step 1:

We reduce Equation (1) to the following nonlinear ordinary differential equation (ODE):

$$\xi = k(x + y + \cdots) - Vt,$$

where t is a positive real number, k is the wave number, and V is the velocity of the traveling wave. We then obtain:

$$Q(U, -VU', kU', kU'', \dots) = 0, \tag{3}$$

where $U(\xi) = u(x, y, t, \dots)$ and Q is a polynomial in U and its derivatives with respect to $\boldsymbol{\xi}$.

Step 2:

Assume that the traveling wave solution of Equation (3) can be expressed as a polynomial of the following form:

$$U(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i + \sum_{i=1}^{N} b_i \left(\frac{G'}{G}\right)^{-i}, \tag{4}$$

where a_i $(i = 0, 1, 2, \dots, N)$ and b_i $(i = 1, 2, \dots, N)$ are constants to be determined, and $G = G(\xi)$ satisfies the Riccati equation:

$$\left(\frac{G'}{G}\right) = \rho \left(\frac{G'}{G}\right)^2 + \mu \left(\frac{G'}{G}\right) + \sigma, \tag{5}$$

where ρ, μ, and σ are arbitrary constants.

Step 3:
Determine the positive integer N by balancing the highest-order nonlinear terms with the highest derivatives in Equation (3).

Step 4:

The general solution of Equation (5) corresponds to one of the following five cases:

$$\frac{G'}{G^2}=\begin{cases}-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}}{2\rho}\left(\frac{A\sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)+B\cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)}{A\cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)+B\sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)}\right), & \text{if } \mu\neq 0, \Delta\geq 0,\\ -\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}}{2\rho}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)+B\cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}{A\cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)+B\sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}\right), & \text{if } \mu\neq 0, \Delta<0,\\ \sqrt{\frac{\sigma}{\rho}}\left(\frac{A\cos(\sqrt{\sigma\rho}\xi)+B\sin(\sqrt{\sigma\rho}\xi)}{-A\sin(\sqrt{\sigma\rho}\xi)+B\cos(\sqrt{\sigma\rho}\xi)}\right), & \text{if } \sigma\rho>0, \mu=0,\\ -\frac{\sqrt{|\sigma\rho|}}{\rho}\left(\frac{A\sinh(2\sqrt{|\sigma\rho|}\xi)+A\cosh(2\sqrt{|\sigma\rho|}\xi)+B}{A\sinh(2\sqrt{|\sigma\rho|}\xi)+A\cosh(2\sqrt{|\sigma\rho|}\xi)-B}\right) & \text{if } \sigma\rho<0, \mu=0,\\ \frac{-A}{\rho(A\xi+B)^2}, & \text{if } \sigma=0, \mu=0, \rho\neq 0.\end{cases} \tag{6}$$

Here, A and B are constants, and $\Delta=\mu^2-4\rho\sigma$.

Step5:
Use Equation (5) to compute the derivatives, then substitute them along with Equation (4) into Equation (3). By collecting all terms with the same powers, form a system of algebraic equations. Solve this system using a software tool such as Maple. Finally, use Equation (6) to obtain the exact solutions of Equation (2).

Application of the G'/G^2 Expansion Method

In this section, the Benjamin–Ono equation, which exhibits various properties in soliton theory, is considered using the effective and practical G'/G^2 expansion method to generate traveling wave solutions. Let us consider the BO equation given by (1).

Assume the transformation:

$$U(x,t)=u(\xi),\quad \xi=x-Vt,$$

Then we find the following derivatives:

$$u_t=-Vu',\quad u_x=u',\quad u_{xx}=u''.$$

Substituting these into Equation (1), we obtain the following ordinary differential equation (ODE):

$$-Vu'+uu'+Hu''=0$$

or equivalently:

$$u'(u-V)+Hu''=0.$$

The highest derivative term is u'' and the highest nonlinear term is u^2, hence at most the square of (G'/G^2) is taken. This implies N = 1.

Therefore, the solution is written as:

$$\mathrm{U}(\xi)=a_0+\sum_{1=1}^{N}\left[a_n\left(\frac{G'}{G^2}\right)^n+b_n\left(\frac{G'}{G^2}\right)^{-n}\right]$$

Here, the function $G = G(\xi)$ satisfies the following relation:

$$\left(\frac{G'}{G^2}\right)' = \sigma + \mu\left(\frac{G'}{G^2}\right) + p\left(\frac{G'}{G^2}\right)^2$$

where μ, σ, p are free parameters. The constants a_0, a_n, b_n must be determined in the solution.

Assume;

$$\mathrm{U}(\xi) = a_0 + a_1 \cdot \left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1}$$

Now compute the necessary derivatives:

$$u' = a_1\left(\frac{G'}{G^2}\right)' + b_1(-1)\cdot\left(\frac{G'}{G^2}\right)^{-2}\left(\frac{G'}{G^2}\right)'$$

$$=a_1\left[\sigma + \mu\left(\frac{G'}{G^2}\right) + p\left(\frac{G'}{G^2}\right)^2\right] - b_1\left(\frac{G'}{G^2}\right)^{-2}\left[\sigma + \mu\left(\frac{G'}{G^2}\right) + P\left(\frac{G'}{G^2}\right)^2\right]$$

$$=a_1\sigma + a_1\mu\left(\frac{G'}{G^2}\right) + a_1 p\left(\frac{G'}{G^2}\right)^2 - b_1\left(\frac{G'}{G^2}\right)^{-2}\sigma - b_1\left(\frac{G'}{G^2}\right)^{-2}\mu\left(\frac{G'}{G^2}\right) - b_1\left(\frac{G'}{G^2}\right)^{-2}p\left(\frac{G'}{G^2}\right)^2$$

$$u'=a_1\sigma + a_1\mu\left(\frac{G'}{G^2}\right) + a_1 p\left(\frac{G'}{G^2}\right)^2 - b_1\sigma\left(\frac{G'}{G^2}\right)^{-2} - b_1\mu\left(\frac{G'}{G^2}\right)^{-1} - b_1 p$$

$$u'' = a_1\mu\left(\frac{G'}{G^2}\right)' + 2a_1 p\left(\frac{G'}{G^2}\right)\left(\frac{G'}{G^2}\right)' + 2b_1\sigma\left(\frac{G'}{G^2}\right)^{-3}\left(\frac{G'}{G^2}\right)' + b_1\mu\left(\frac{G'}{G^2}\right)^{-2}\left(\frac{G'}{G^2}\right)'$$

Substituting $(G'/G^2)'$ from earlier;

$$=a_1\mu\left[\sigma + \mu\left(\frac{G'}{G^2}\right) + p\left(\frac{G'}{G^2}\right)^2\right] + 2a_1 p\left(\frac{G'}{G^2}\right)\left[\sigma + \mu\left(\frac{G'}{G^2}\right) + p\left(\frac{G'}{G^2}\right)^2\right] + 2b_1\sigma\left(\frac{G'}{G^2}\right)^{-3}\left[\sigma + \mu\left(\frac{G'}{G^2}\right) + p\left(\frac{G'}{G^2}\right)^2\right] + b_1\mu\left(\frac{G'}{G^2}\right)^{-2}\left[\sigma + \mu\left(\frac{G'}{G^2}\right) + p\left(\frac{G'}{G^2}\right)^2\right]$$

This simplifies to:

$$u''=a_1\mu\sigma + a_1\mu^2\left(\frac{G'}{G^2}\right) + a_1\mu p\left(\frac{G'}{G^2}\right)^2 + 2a_1 p\sigma\left(\frac{G'}{G^2}\right) + 2a_1 p\mu\left(\frac{G'}{G^2}\right)^2 + 2a_1 p^2\left(\frac{G'}{G^2}\right)^3 + 2b_1\sigma^2\left(\frac{G'}{G^2}\right)^{-3} + 2b_1\sigma\mu\left(\frac{G'}{G^2}\right)^{-2} + 2b_1\sigma p\left(\frac{G'}{G^2}\right)^{-1} + b_1\mu\sigma\left(\frac{G'}{G^2}\right)^{-2} + b_1\mu^2\left(\frac{G'}{G^2}\right)^{-1} + b_1\mu p$$

Substitute these expressions into the equation:

$$u'(u - V) + H u'' = 0,$$

$$\left[a_1\sigma + a_1\mu\left(\frac{G'}{G^2}\right) + a_1 p\left(\frac{G'}{G^2}\right)^2 - b_1\sigma\left(\frac{G'}{G^2}\right)^{-2} - b_1\mu\left(\frac{G'}{G^2}\right)^{-1} - b_1 p\right]\left[a_0 + a_1\cdot\left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1} - V\right] + H\left[a_1\mu\sigma + a_1\mu^2\left(\frac{G'}{G^2}\right) + a_1\mu p\left(\frac{G'}{G^2}\right)^2 + 2a_1 p\sigma\left(\frac{G'}{G^2}\right) + 2a_1 p\mu\left(\frac{G'}{G^2}\right)^2 + 2a_1 p^2\left(\frac{G'}{G^2}\right)^3 + 2b_1\sigma^2\left(\frac{G'}{G^2}\right)^{-3} + 2b_1\sigma\mu\left(\frac{G'}{G^2}\right)^{-2} + 2b_1\sigma p\left(\frac{G'}{G^2}\right)^{-1} + b_1\mu\sigma\left(\frac{G'}{G^2}\right)^{-2} + b_1\mu^2\left(\frac{G'}{G^2}\right)^{-1} + b_1\mu p\right] = 0$$

$$a_0a_1\sigma + a_1^2\sigma\left(\frac{G'}{G^2}\right) + a_1b_1\sigma\left(\frac{G'}{G^2}\right)^{-1} - a_1\sigma V + a_0a_1\mu\left(\frac{G'}{G^2}\right) + a_1^2\mu\left(\frac{G'}{G^2}\right)^2 + a_1b_1\mu - a_1\mu V\left(\frac{G'}{G^2}\right)$$
$$+ a_0a_1p\left(\frac{G'}{G^2}\right)^2 + a_1^2p\left(\frac{G'}{G^2}\right)^3 + a_1b_1p\left(\frac{G'}{G^2}\right) - a_1pV\left(\frac{G'}{G^2}\right)^2 - a_0b_1\sigma\left(\frac{G'}{G^2}\right)^{-2}$$
$$- a_1b_1\sigma\left(\frac{G'}{G^2}\right)^{-1} - b_1^2\sigma\left(\frac{G'}{G^2}\right)^{-3} + b_1\sigma V\left(\frac{G'}{G^2}\right)^{-2} - a_0b_1\mu\left(\frac{G'}{G^2}\right)^{-1} - a_1b_1\mu - b_1^2\mu\left(\frac{G'}{G^2}\right)^{-2}$$
$$+ b_1\mu V\left(\frac{G'}{G^2}\right)^{-1} - a_0b_1p - a_1b_1p\left(\frac{G'}{G^2}\right) - b_1^2p\left(\frac{G'}{G^2}\right)^{-1} + b_1Vp + Ha_1\mu\sigma + Ha_1\mu^2\left(\frac{G'}{G^2}\right)$$
$$+ Ha_1\mu p\left(\frac{G'}{G^2}\right)^2 + 2Ha_1p\sigma\left(\frac{G'}{G^2}\right) + 2Ha_1p\mu\left(\frac{G'}{G^2}\right)^2 + 2Ha_1p^2\left(\frac{G'}{G^2}\right)^3 + 2Hb_1\sigma^2\left(\frac{G'}{G^2}\right)^{-3}$$
$$+ 2Hb_1\sigma\mu\left(\frac{G'}{G^2}\right)^{-2} + 2Hb_1\sigma p\left(\frac{G'}{G^2}\right)^{-1} + Hb_1\mu\sigma\left(\frac{G'}{G^2}\right)^{-2} + Hb_1\mu^2\left(\frac{G'}{G^2}\right)^{-1} + Hb_1\mu p = 0$$

The following system of equations can be constructed by taking the necessary derivatives, a_0 , a_1 ,b_1, σ, μ,σ, p are arbitrary constants, making the necessary adjustments and setting the coefficients of like powers of (G′/G²) and set them equal to zero, leading to the following algebraic system:

$$\left(\frac{G'}{G^2}\right)^3 = a_1^2p + 2Ha_1p^2 = 0$$

$$\left(\frac{G'}{G^2}\right)^2 = a_1^2\mu + a_1pa_0 - a_1pV + 3Ha_1p\mu = 0$$

$$\left(\frac{G'}{G^2}\right) = a_1^2\sigma + a_0a_1\mu - a_1\mu V + Ha_1\mu^2 + 2Ha_1p\sigma = 0$$

$$\left(\frac{G'}{G^2}\right)^{-1} = -a_0b_1\mu + b_1\mu V - b_1^2p + 2Hb_1\sigma p + Hb_1\mu^2 = 0$$

$$\left(\frac{G'}{G^2}\right)^{-2} = -a_0b_1\sigma + b_1V\sigma - b_1^2\mu + 3Hb_1\sigma\mu = 0$$

$$\left(\frac{G'}{G^2}\right)^{-3} = -b_1^2\sigma + 2Hb_1\sigma^2 = 0$$

This algebraic system can be solved using symbolic computation software, we obtain:

$$\mathrm{a}_0 = -\frac{V+p\mu^2+3p\sigma}{2H},\ \mathrm{a}_1 = -\frac{p\mu}{H}, \mathrm{b}_1 = -\frac{2p}{H},$$

Consequently, we have the following different cases for the exact solutions of Benjamin Ono equation:

Case 1: When $\Delta = \mu^2 - 4p\sigma > 0, \mu \neq 0,$

$$u_1(x,t) = -\frac{V+p\mu^2+3p\sigma}{2H} + \frac{2p\mu}{H(\sqrt{\mu^2-4p\sigma}}\frac{1}{\tanh\left(\frac{\sqrt{\mu^2-4p\sigma}}{2}\xi\right)} + \mu - \frac{8p}{H(\sqrt{\mu^2-4\rho\sigma}}\frac{1}{\tanh\left(\frac{\sqrt{\mu^2-4p\sigma}}{2}\xi\right)} + \mu$$

Case 2: When $\Delta = \mu^2 - 4p\sigma < 0, \mu \neq 0,$

$$u_2(x,t) = -\frac{\mathrm{V}+p\mu^2+3p\sigma}{2\mathrm{H}} + \frac{2p\mu}{H\left(-\sqrt{4p\sigma-\mu^2}\right)}\frac{1}{\tan\left(\frac{\sqrt{4p\sigma-\mu^2}}{2}\xi\right)} + \mu - \frac{8p}{H(\sqrt{\mu^2-4\rho\sigma})}\frac{1}{\tan\left(\frac{\sqrt{4p\sigma-\mu^2}}{2}\xi\right)} + \mu.$$

III. RESULTS

In this study, the Benjamin–Ono equation was analyzed using the (G'/G^2) expansion method to obtain analytical traveling wave solutions. By transforming the equation into an ordinary differential form and applying the method systematically, two distinct sets of parameter values were identified that yield exact solutions. These solutions were expressed in terms of hyperbolic and trigonometric functions such as tanh, tan, and sec, depending on the case. The coefficients were calculated by solving the resulting algebraic system, and the corresponding wave solutions were explicitly derived.

IV.DISCUSSION

The obtained solutions describe solitary and periodic wave structures, which are consistent with the known physical behavior of internal waves in deep fluids, modeled by the Benjamin–Ono equation. The two cases explored in the study correspond to different types of wave motion:The first solution involving tanh represents a solitary wave (soliton), characterized by a localized, non-periodic waveform that maintains its shape while traveling.The second solution involving tan and sec corresponds to periodic wave behavior, suggesting wave trains or repeating patterns.The results are in agreement with similar analytical methods reported in the literature, such as the tanh-method or the extended Riccati equation approach.

V. CONCLUSION

This study successfully applied the (G'/G^2) expansion method to the Benjamin–Ono equation, providing new exact solutions that reveal important wave characteristics. The method proved to be effective, systematic, and flexible, highlighting its potential in solving nonlinear evolution equations with physical significance. By offering analytical insight into wave behavior, the study not only confirms existing findings but also contributes to expanding the known solution space of the Benjamin–Ono equation. Future research may apply the same method to other integrable and non-integrable equations to further explore nonlinear wave dynamics.

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