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Research Article

Application of dynamic analysis in systems with MDOF and distributed parameters using MathCad software

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Abstract **–** Dynamic analysis plays a fundamental role in evaluating the response of structural systems subjected to time-dependent actions such as seismic, wind, or impact loads. This study focuses on the dynamic characterization of structures through the use of the distribution parameter MDF, which is employed to describe the contribution and spatial distribution of dynamic effects within the system. The MDF parameter is formulated based on modal properties and governing dynamic equilibrium equations, allowing the interaction between mass, and stiffness, to be consistently represented. A systematic analytical framework is presented to evaluate the sensitivity of the dynamic response to variations in MDF, highlighting its influence on displacement, acceleration, and internal force demands. The proposed approach enables an efficient interpretation of modal participation and load distribution under dynamic excitation, offering improved accuracy compared to conventional simplified methods. The results demonstrate that MDF is a key parameter for capturing the dynamic behavior of complex structural systems and can be effectively used to support performance-based design and assessment.

Keywords – Dynamic Analysis, Distributed Mass, İnfinite -DOF, Time History, Response.

I. INTRODUCTION

A major part of dynamic analysis is focused on discretized systems, with lumped masses. These systems are an assemblage of rigid elements with mass (the floor diaphragms of a multistory building) and massless elements that are flexible (the beams and columns of a building). Many classes of structures (e.g., multistory buildings) can effectively idealize like lumped- mass discretized systems. Effective methods are ideal for computer implementation to solve the ordinary differential equations systems for these systems, also.

However, a lumped- mass idealization, although applicable, is not a natural approach for certain types of structures, such as a chimney, an arch dam, or a nuclear containment structure.

The structural dynamics problem for one- dimensional systems with distributed mass, such as a tower, and solutions are presented for simple system in this paper. The solutions presented for this simple case provides insight into the dynamics of distributed – mass systems which have an infinite number of DOFs and how this differs from lumped- mass systems with a finite number of DOFs.

II. METHOD

2.1 Equation of undamped motion

A beam with flexural rigidity EI(x) and mass m(x) per unit length (both of which may vary with position x) we develop the motion equation governing the transverse vibration without damping subjected to external force p (x, t), and transverse displacement u (x,t).

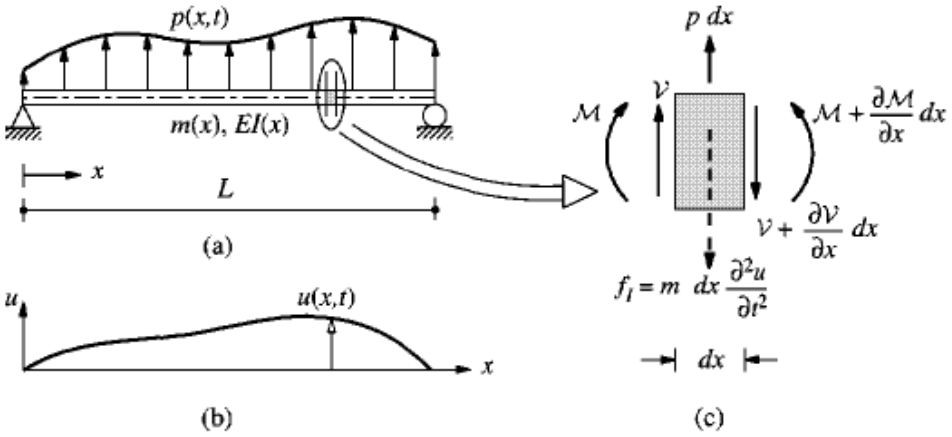


Figure 1. System with distributed mass and elasticity: a) beam and external force b) displacement c) forces on element ([1] Chopra A., pg. 586)

The equation of motion developed will be valid for support conditions other than the simple supports.
The system has an infinite number of DOSs, because its mass is distributed. The forces on the element (Fig. 1, c), where inertia force has been included according to D'Alembert' s principle V (x, t) is the transverse shear force and M (x, t) is the bending moment. In y- direction the equilibrium of forces is given by:

$$\frac{\partial V}{\partial x} = p - m\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

The inertia force modifies the external force in recognition of the dynamic of the problem. If the inertial moment associated with angular acceleration of the element is neglected, rotational equilibrium of the element gives the standard relation:

$$V = \frac{\partial M}{\partial x} \qquad (2)$$

To write the equation of u(x,t), using relations 1, 2, with shear deformation neglected, the moment-curvature relation is:

$$M = EI\frac{\partial^2 u}{\partial x^2} \qquad (3)$$

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = p(x,t) \qquad (4)$$

This is the partial differential equation governing the motion u(x,t) of the beam subjected to external dynamic forces p(x,t). To obtain a unique solution, must be specified boundary conditions, initial displacement u (x,0) and initial velocity u'(x,0).

2.2 Equation of undamped motion: support excitation

A cantilever beam subjected to horizontal base motion or a beam with multiple supports subjected to identical motion in the vertical direction, the total displacement is:

$$u^t(x,t) = u(x,t) + u_g(t) \qquad (5)$$

u (x, t)- beam displacement, measured relative to the support motion $u_g(t)$, results from the deformations of the beam.

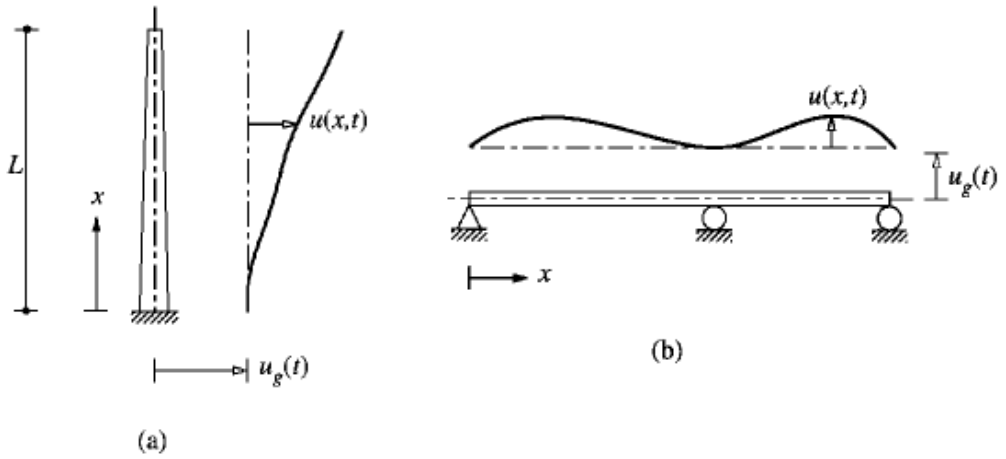


Figure 2. a) Cantilever beam subjected to base excitation; b) continuous beam subjected to identical motion at all supports

For these simple cases of a beam excited by support motion the derivation of the equation of motion is only slightly different than that for applied forces. Recognizing that the inertia forces are now related to the total accelerations and that external forces p(x,t) do not exist, eq. 1) becomes:

$$\frac{\partial V}{\partial x} = -m\frac{\partial^2 u'}{\partial t^2} = -m\frac{\partial^2 u}{\partial t^2} - m\frac{d^2 u_g}{dt^2} \qquad (6)$$

Wherein eq. (5) has been used to obtain the second half of the equation. Substituting Eq. (3) and (2) into Eq. (6) gives:

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = -m(x)u_g^{"}(t) \qquad (7)$$

By comparing Eqs. (7) and (1), it is clear that the deformation response u(x,t) of the beam to support acceleration $u"_g(t)$ will be identical to the response of the system with stationary support due to external forces = $-m(x)u"_g(t)$. The support excitacion can therefore be replaced by effective forces (Fig.3):

$$p_{eff}(x,t) = -m(x)\,u"_g(t) \qquad (8)$$

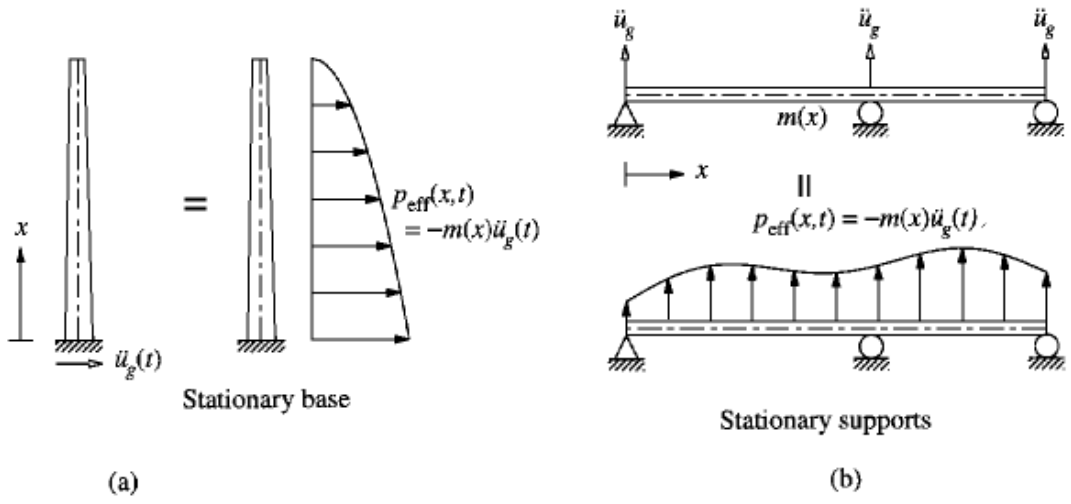


Figure 3. Effective forces p_{eff} (x,t)

This formulation can be generalized to include the possibility of different motions of the various supports of structure. Such multiple support excitation may exist in several practical situations, because it is usually not possible to analyze such practical problems as infinite -DOF systems. They are usually discretized by the finite element method FEM.

2.3 Natural vibration frequencies and modes

For the case of free vibration, Eq. (4) and (7) become:

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = 0 \qquad (9)$$

We attempt a solution of the form

$$u(x,t) = f(x)q(t) \qquad (10)$$

then $$\frac{\partial^2 u}{\partial t^2} = \phi(x)\ddot{q}(t) \qquad \frac{\partial^2 u}{\partial x^2} = \ddot{\phi}(x)q(t) \qquad (11)$$

where over dots denote a time derivative and primes denote an x-derivative;

thus $q'(t)=\partial q/\partial t$, $q''(t)= \partial^2 q/\partial t^2$, and $f'(x) = \partial^2 f/\partial x^2$. Substituting Eq. (11) to Eq.(9) leads to

$$m(x)f(x)q''(t)+q(t)[EI(x)f''(x)]''=0$$

which, when divided by m(x)f(x)q(t), becomes:

$$\frac{-\ddot{q}(t)}{q(t)} = \frac{[EI(x)\ddot{\phi}(x)]''}{m(x)\phi(x)} \qquad (12)$$

The expression on the left is a function of *t* only and the one the right depends only on *x*. For Eq. (12) to be valid for all values of x and t, the two expressions must be constant, say w^2. Thus, the partial differential equation (12) becomes two ordinary differential equations, one governing the time function q(t) and the other governing the spatial function f(x):

$$\ddot{q} + \omega^2 q = 0 \quad (13)$$

$$[EI(x)f''(x)]'' - w^2\, m(x)f(x) = 0 \qquad (14)$$

Eq. (13) has the same form as equation governing free vibration of an SDF system with natural frequency w. For any given stiffness and mass functions, EI(x) and m(x), respectively, there is an infinite set of frequencies w and associated modes f(x) that satisfy the eigenvalue problem defined by Eq. (14) and the boundary conditions for the beam.

For the special case of a uniform beam, EI(x)= EI and m(x)=m Eq. (14) becomes

$$EI(x)f^{IV}(x) - w^2\, m\, f(x) = 0 \quad \text{or} \quad f^{IV}(x) - b^4\, f(x) = 0 \qquad (15)$$

Where: $$b^4 = w^2 m / EI \qquad (16)$$

The general solution of Eq. 15 is:

$$f(x) = C_1 \sin bx + C_2 \cos bx + C_3 \sinh bx + C_4 \cosh bx \qquad (17)$$

This solution contains four unknown constants, C_1, C_2, C_3, C_4 and the eigenvalue parameter b. Application of the four boundary conditions for a single – span beam, two at each end of the beam, will provide a solution for b and hence for the natural frequency w from Eq.(16).

For uniform cantilever beam the natural vibration frequencies and modes are determined: At the clamped end,

$$u(0) = 0 \rightarrow f(0) = 0 \rightarrow C_2 + C_4 = 0 \rightarrow C_4 = -C_2 \qquad (18a)$$

$$u'(0) = 0 \rightarrow f'(0) = 0 \rightarrow b(C_1+C_3) = 0 \rightarrow C_3 = -C_1 \quad (18b)$$

At the free end x=L of the cantilever moment and shear are both zero. Thus, from Eq. (17) and after using Eq. (18), we obtain:

$M(L)=0 \rightarrow EI\, f''(L)=0$

$$\rightarrow C_1(\sin bL + \sinh bL) + C_2(\cos bL + \cosh bL) = 0 \quad (19)$$

$V(L)=0 \rightarrow EI\, f'''(L)=0$

$$\rightarrow C_1(\cos bL + \cosh bL) + C_2(-\sin bL + \sinh bL) = 0 \quad (20)$$

- We can find the natural vibration modes by equation:

$$\Phi_n(x) = \cosh\beta_n x - \cos\beta_n x - \frac{\cosh\beta_n L + \cos\beta_n L}{\sinh\beta_n L + \sin\beta_n L}(\sinh\beta_n x - \sin\beta_n x) \quad (21)$$

III. RESULTS

Applications of the theory of dynamic analysis for distribution parameter MDF is to determine the first three natural frequencies and corresponding modal shapes of three cells; to find the response of displacement, acceleration as well as the shear forces and bending moments in 10 points along the height of the silo. Input motion is the EL Centro earthquake. And, in the end to plot the response time histories for the top level. MB= 30 MPa.

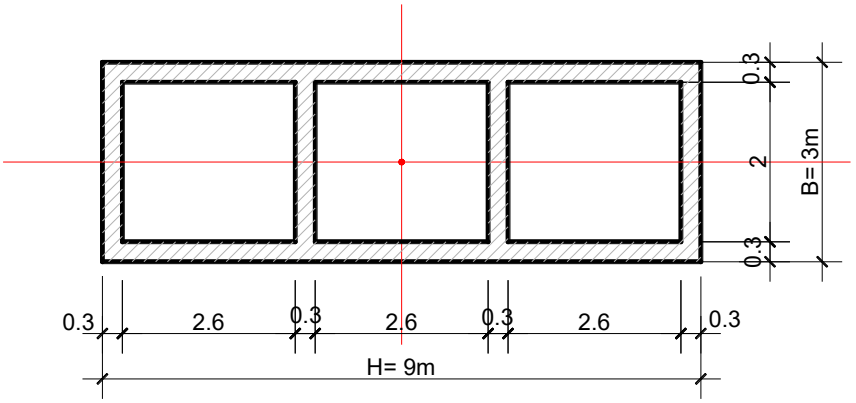


Figure 4. Section of the silo

$$m(x) = \bar{m} = \frac{\gamma \cdot F \cdot 1m}{g} = \frac{2400daN/m^3 \cdot 8.28m^2 \cdot 1m}{9.81m/s^2} = 2026daN/m$$

$F=9\cdot3-3\cdot2{,}6\cdot2{,}4= 8{,}28\ m^2$

$$I_y = \frac{9\cdot3^3}{12} - 3\cdot\frac{2.6\cdot2.4^3}{12} = 11.2644m^4 \qquad EI_y = 339\cdot10^5\ daN\cdot m^2$$

$$I_z = \frac{3\cdot9^3}{12} - \frac{2.4\cdot2.6^3}{12} - 2\left[\frac{2.4\cdot2.6^3}{12} + 2.9^2(2.6\cdot2.4)\right] = 66.7476m^4 \qquad EI_z = 2001\cdot10^5\ daN\cdot m^2$$

Natural frequencies for the cantilever beam with distributed parameters

Direction I_y

$$\omega_1 = \frac{3.516}{L^2}\sqrt{\frac{EI_y}{m}} = \frac{3.516}{15^2}\sqrt{\frac{339\cdot10^5}{2026}} = 2.02(1/s) \qquad T_1 = \frac{2\pi}{\omega_1} = \frac{2\cdot3.14}{2.02} = 3.1s$$

$$\omega_2 = \frac{22.03}{L^2}\sqrt{\frac{EI_y}{m}} = \frac{22.03}{15^2}\sqrt{\frac{339\cdot10^5}{2026}} = 12.66(1/s) \qquad T_2 = \frac{2\pi}{\omega_2} = \frac{2\cdot3.14}{12.66} = 0.49s$$

$$\omega_3 = \frac{61.70}{L^2}\sqrt{\frac{EI_y}{m}} = \frac{61.70}{15^2}\sqrt{\frac{339\cdot10^5}{2026}} = 35.47(1/s) \qquad T_3 = \frac{2\pi}{\omega_3} = \frac{2\cdot3.14}{35.47} = 0.18s$$

Direction I_z

$$\omega_1 = \frac{3.516}{L^2}\sqrt{\frac{EI_z}{m}} = \frac{3.516}{15^2}\sqrt{\frac{2001\cdot10^5}{2026}} = 4.91(1/s) \qquad T_1 = \frac{2\pi}{\omega_1} = \frac{2\cdot3.14}{4.91} = 1.28s$$

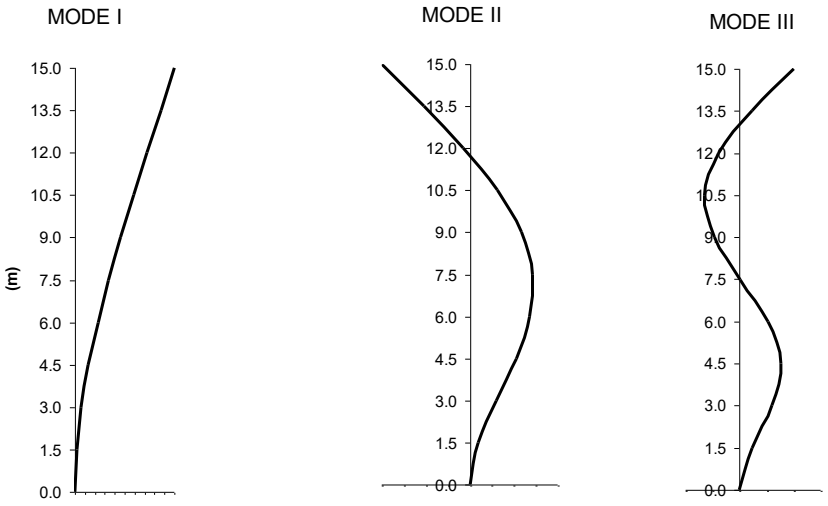
$$\omega_2 = \frac{22.03}{L^2}\sqrt{\frac{EI_z}{m}} = \frac{22.03}{15^2}\sqrt{\frac{2001\cdot10^5}{2026}} = 30.77(1/s) \qquad T_2 = \frac{2\pi}{\omega_2} = \frac{2\cdot3.14}{30.77} = 0.2s$$

$$\omega_3 = \frac{61.70}{L^2}\sqrt{\frac{EI_z}{m}} = \frac{61.70}{15^2}\sqrt{\frac{2001\cdot10^5}{2026}} = 86.18(1/s) \qquad T_3 = \frac{2\pi}{\omega_3} = \frac{2\cdot3.14}{86.18} = 0.073s$$

- We can find the natural vibration modes by equation:

$$\Phi_n(x) = \cosh\beta_n x - \cos\beta_n x - \frac{\cosh\beta_n L + \cos\beta_n L}{\sinh\beta_n L + \sin\beta_n L}(\sinh\beta_n x - \sin\beta_n x)$$

MODE SHAPES



Input motion is EL-Centro earthquake. We chose u_g" max from the interval of 12s u_g"=0.3194 m/s^2 We find first the displacement for the three modes in direction of I_y, then total displacement in 10 points of the silo.

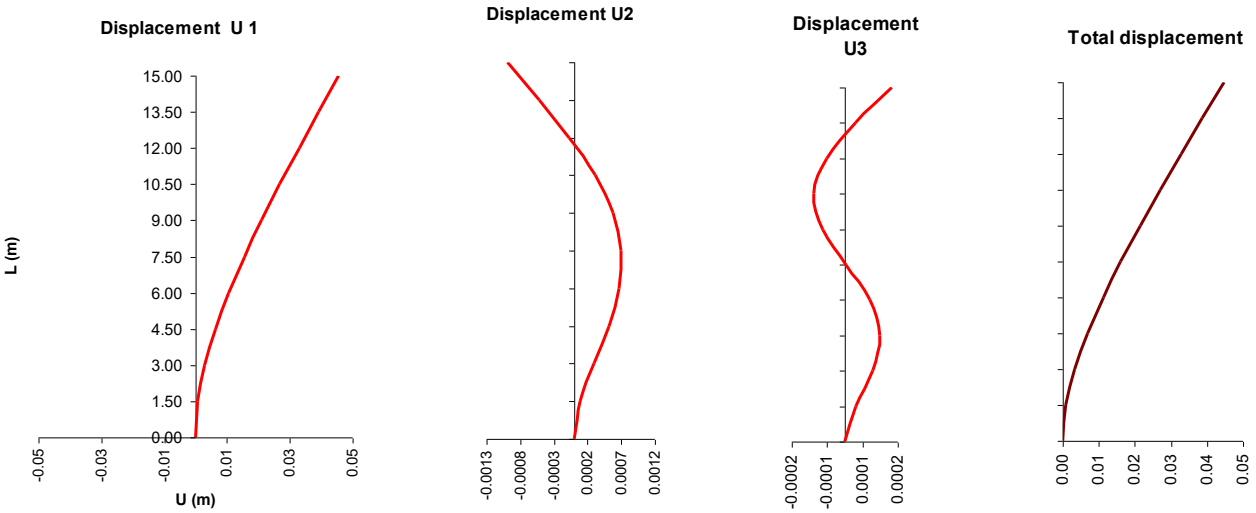
$$U_{total} = \sum_{n=1}^{3} u_n \qquad Q_{total} = \sum_{n=1}^{3} Q_n \qquad M_{total} = \sum_{n=1}^{3} M_n$$

$$u_n = \Phi_n \frac{1}{\omega_n} \frac{\int_0^L \Phi_n(x)dm}{\int_o^L \Phi_n^2(x)dm} \int_0^t u_g''(\tau) \sin \omega_n (t-\tau)d\tau$$

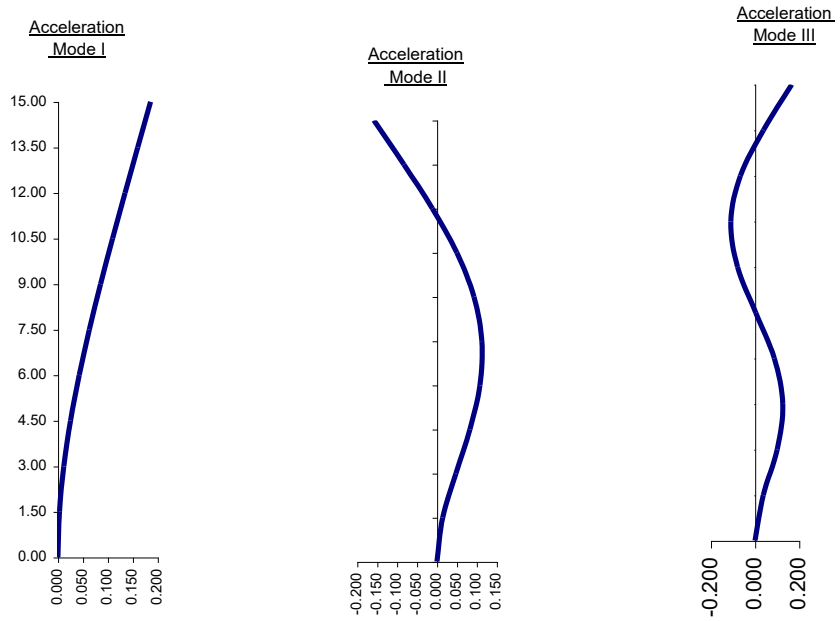
$$M_n = EI_y \frac{d^2\Phi_n(x)}{dx^2} \frac{1}{\omega_n} \frac{\int_0^L \Phi_n(x)dm}{\int_o^L \Phi_n^2(x)dm} \int_0^t u_g''(\tau) \sin \omega_n (t-\tau)d\tau$$

$$Q_n = EI_y \frac{d^3\Phi_n(x)}{dx^3} \frac{1}{\omega_n} \frac{\int_0^L \Phi_n(x)dm}{\int_o^L \Phi_n^2(x)dm} \int_0^t u_g''(\tau) \sin \omega_n (t-\tau)d\tau$$

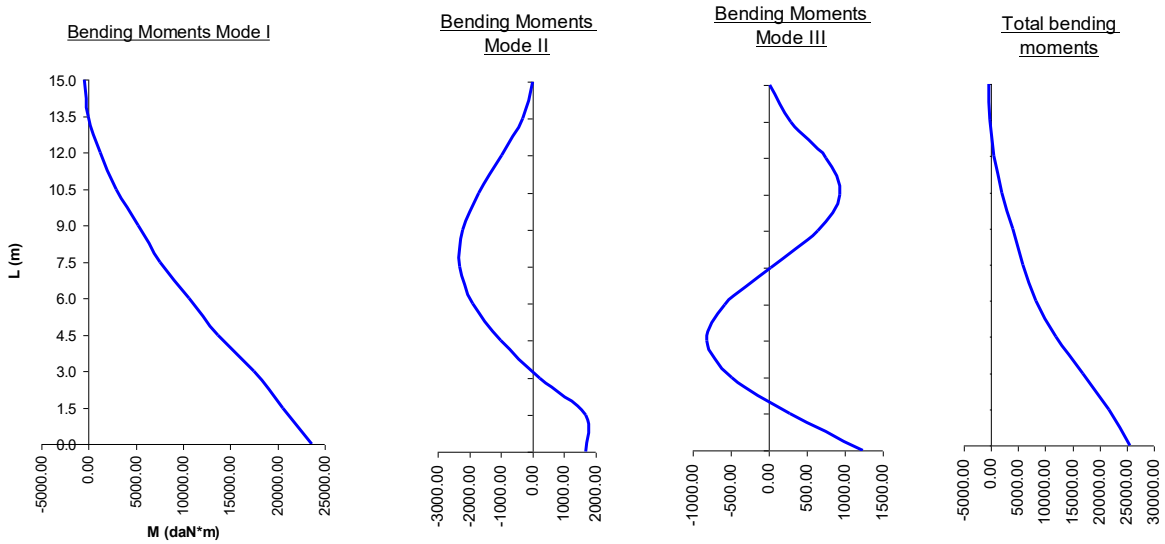
DISPLACEMENT (m)



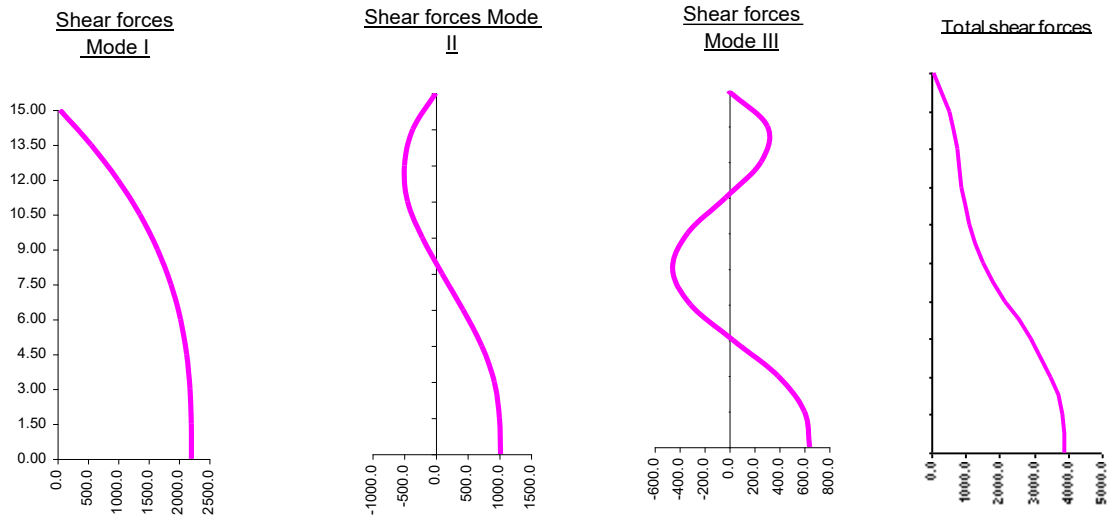
ACCELERATIONS (m/ s^2)



BENDING MOMENTS (daN·m)



SHEARS (daN)



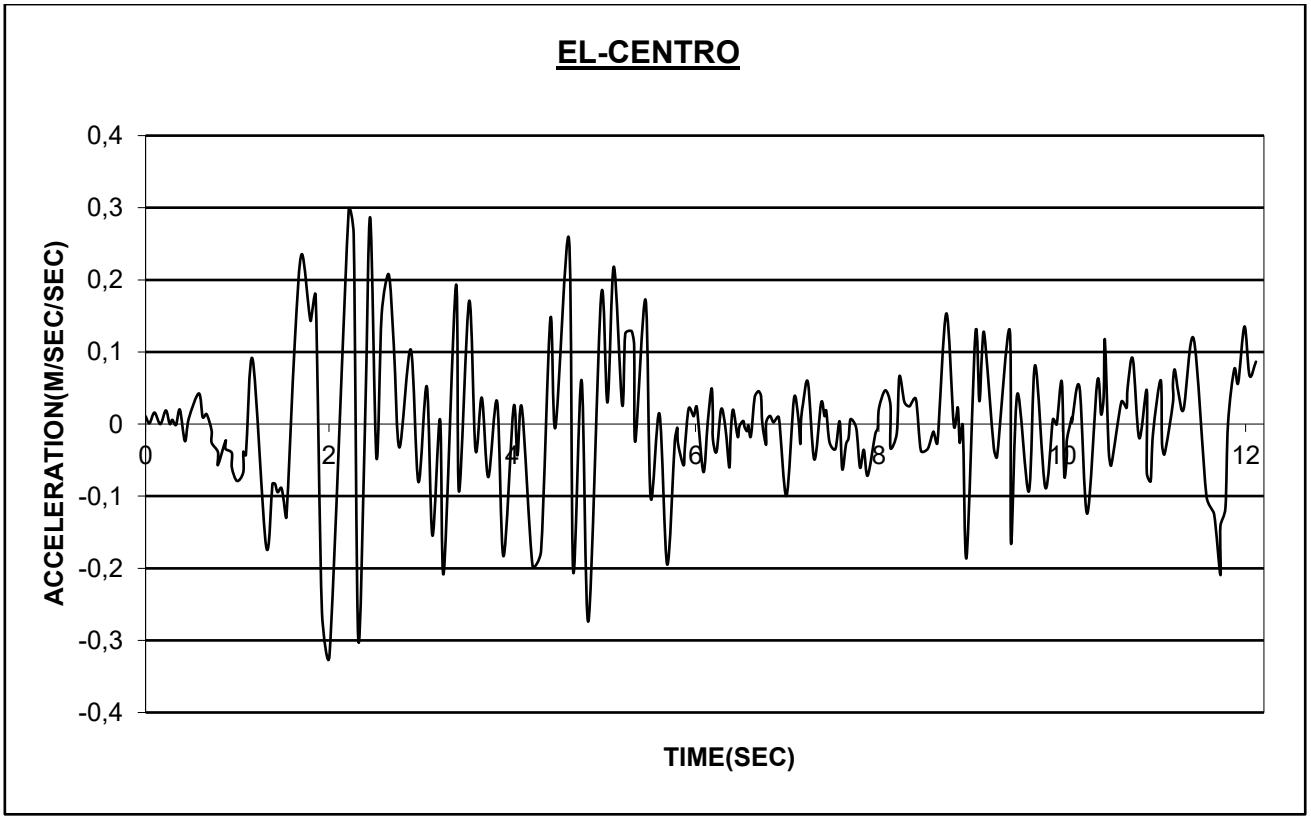
Mode	Points	x(m)	L(m)	m(x) (kg/ml)	EI_y	$\square_1(x)$	u_1 (m)	M_1(daN*m)	Q_1(daN)	u_1'' (m/s^2)	U_{total}
I	0	0.00	15	2026	$33.9*10^6$	0.0000	0.0000	23579.47	2208.10	0.000	0.0000
I	1	1.50	15	2026	$33.9*10^6$	0.0335	0.0008	20536.96	2205.06	0.003	0.0009
I	2	3.00	15	2026	$33.9*10^6$	0.1277	0.0029	17494.44	2183.76	0.012	0.0033
I	3	4.50	15	2026	$33.9*10^6$	0.2730	0.0062	13691.30	2129.00	0.025	0.0068
I	4	6.00	15	2026	$33.9*10^6$	0.4598	0.0105	10648.79	2027.83	0.043	0.0113
I	5	7.50	15	2026	$33.9*10^6$	0.6790	0.0155	7606.28	1870.38	0.063	0.0162
I	6	9.00	15	2026	$33.9*10^6$	0.9223	0.0210	5220.95	1649.04	0.086	0.0215
I	7	10.50	15	2026	$33.9*10^6$	1.1817	0.0269	2957.32	1357.72	0.110	0.0271
I	8	12.00	15	2026	$33.9*10^6$	1.4510	0.0330	1184.30	993.38	0.135	0.0329

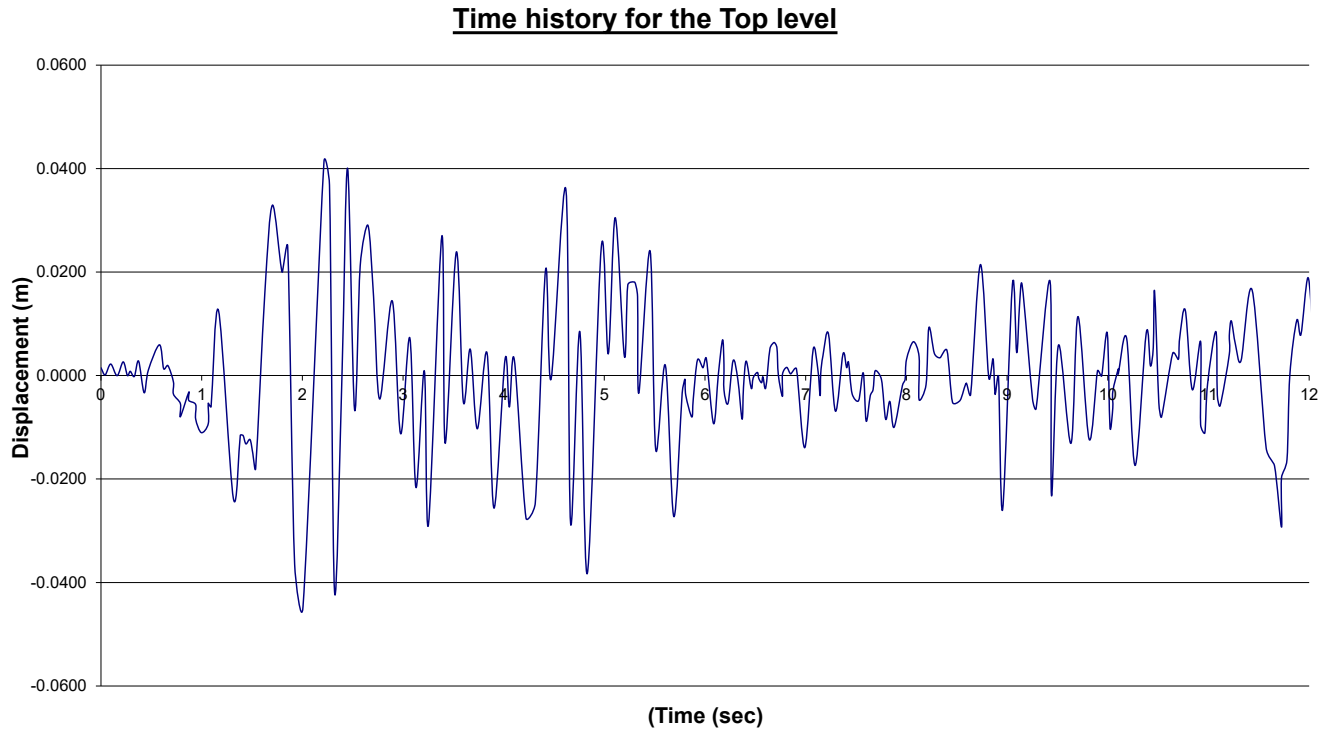
I	9	13.50	15	2026	$33.9*10^6$	1.7248	0.0393	13.43	554.88	0.160	0.0388
I	10	15.00	15	2026	$33.9*10^6$	2.0000	0.0455	-44.28	40.94	0.186	0.0446

Mode	Points	x	L	m(x) (kg/ml)	EI_y	□□**(x)**	**u_2 (m)**	**M_2**	**Q_2 (daN)**	**u_2"**	**M_{total}**
II	0	0.00	15	2026	$33.9*10^6$	0.0000	0.0000	1696.20	1021.02	0.000	26507.12
II	1	1.50	15	2026	$33.9*10^6$	0.1853	0.0001	1679.74	1004.55	0.016	22497.08
II	2	3.00	15	2026	$33.9*10^6$	0.6021	0.0003	230.55	922.21	0.048	17238.25
II	3	4.50	15	2026	$33.9*10^6$	1.0523	0.0005	-1021.02	724.59	0.080	11860.54
II	4	6.00	15	2026	$33.9*10^6$	1.3669	0.0007	-1893.82	428.17	0.112	8171.77
II	5	7.50	15	2026	$33.9*10^6$	1.4273	0.0007	-2305.52	97.69	0.112	5325.43
II	6	9.00	15	2026	$33.9*10^6$	1.1789	0.0006	-2206.71	214.08	0.096	3662.48
II	7	10.50	15	2026	$33.9*10^6$	0.6341	0.0003	-1696.20	444.64	0.048	2194.23
II	8	12.00	15	2026	$33.9*10^6$	-0.1401	-0.0001	-971.61	494.04	-0.016	959.63
II	9	13.50	15	2026	$33.9*10^6$	-1.0475	-0.0005	-296.42	362.30	-0.080	6.36
II	10	15.00	15	2026	$33.9*10^6$	-2.0000	-0.0010	-2.83	0.30	-0.160	-27.66

Mode	Points	x	L	m(x) (kg/ml)	EI_y	□□**(x)**	**u_3 (m)**	**M_3**	**Q_3 (daN)**	**u_3"**	**Q_{total}**
III	0	0.00	15	2026	$33.9*10^6$	0.0000	0.00000	1231.44	643.76	0.000	3872.88
III	1	1.50	15	2026	$33.9*10^6$	0.4561	0.00003	280.38	601.14	0.038	3810.75
III	2	3.00	15	2026	$33.9*10^6$	1.2090	0.00008	-486.75	388.05	0.101	3494.02
III	3	4.50	15	2026	$33.9*10^6$	1.5125	0.00010	-809.75	29.16	0.126	2882.75
III	4	6.00	15	2026	$33.9*10^6$	1.0518	0.00007	-583.20	-311.79	0.088	2144.22
III	5	7.50	15	2026	$33.9*10^6$	0.0393	0.00000	24.67	-455.34	0.003	1512.73
III	6	9.00	15	2026	$33.9*10^6$	-0.9476	-0.00006	648.25	-334.22	0.079	1528.91
III	7	10.50	15	2026	$33.9*10^6$	-1.3148	-0.00009	933.12	-29.16	-0.109	1773.20
III	8	12.00	15	2026	$33.9*10^6$	-0.7897	-0.00005	746.94	253.47	-0.066	1740.89
III	9	13.50	15	2026	$33.9*10^6$	0.4571	0.00003	289.36	305.06	0.038	1222.23
III	10	15.00	15	2026	$33.9*10^6$	2.0000	0.00013	19.45	10.66	0.166	51.91

- Time history (displacement) for the top level for EL- Centro earthquake





IV. DISCUSSION

The discussion is, why this infinite- DOF approach is not feasible for practical systems, pointing to the need for discretised methods for distributed- mass systems. The results presented for the simple systems provide the exact solution against which results from discretised methods can be compared.

The use of the Finite Element Method in dynamic analysis, through design software, gives a very good overview of the stressed state under the action of seismic force, both with spectral analysis and with Time history. The aim of this presentation was to use the exact solution for dynamic analysis.

V. CONCLUSION

Through the solution of this example, the dynamic analysis of systems with multiple degrees of freedom is illustrated. Dynamic analysis of multi-degree-of-freedom systems offers a very accurate solution. But for this, a rich set of mathematical solutions is necessary. Mathcad software makes very easy the application of long mathematical formulas and you can do variety of charts.

This article does not present all the tables and calculations of integrals, but really it takes a lot of time and many pages of calculus.

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