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# NEW INERTIAL IMPLICIT PROJECTION METHOD FOR SOLVING QUASI-VARIATIONAL INEQUALITIES IN REAL HILBERT SPACES

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***Abstract*** **–** This paper introduces a new inertial implicit projection method to solve quasivariational inequalities with strongly monotone and Lipschitz continuous operators in real Hilbert spaces. We analysed the convergence of a method with varying stepsizes under suitable conditions and also discussed the complexity bound of the proposed algorithm.

*Keywords – Quasi-Variational İnequality; Inertial Extrapolation Step; Strong Monotonicity; Inertial Methods; Convergence.*

## 1. INTRODUCTION

Variational inequalities are essential for solving problems related to mechanics, optimization, transportation, economics, elasticity, etc.. Due to their applications, variational inequalities have been used in many ways. Here we study about quasivariational inequality(QVI) problem, which is extension of classical variational inequality problem of Fichera [17] and Stampacchia [35]. For more details on variational inequalities, quasi-variational inequalities and their applications, we refer to [18,23,26,29,31].

Quasi-variational inequality states that the feasible set of a problem changes according to an explicit or implicit rule. For example, in many applications, the feasible set is defined as a 'moving set' with a closed and convex core set replaced by a single-valued mapping: see for example [1, 24, 28, 29, 31]. In such a setting, the problem is often called 'moving set' quasi-variational inequality. Quasi-variational inequalities are used to model various problems in pure and applied sciences, and Bensoussan and Lions [9] have shown that impulse control problems can be formulated as quasi-variational inequality problems. Quasi-variational inequalities benefit from cross-fertilization between functional analysis, convex analysis, numerical analysis and physics. From these interactions so many numerical techniques have been developed to solve quasi-variational inequalities and optimization problems, see[ $14-2022$ ]and references therein.

Quasi-variational inequalities can be solved using various techniques; for example, very recent,Antipin et al. [11] developed gradient projection and extragradient methods for solving quasi-variational inequalities. The main disadvantage of the extragradient method with respect to the classical gradient method, is that it has a doubled number of orthogonal projections and mapping evaluations per iteration. Meanwhile in the context of variational inequalities, extragradient method guarantees convergence under weaker assumptions than strong monotonicity of associated mapping, but extragradient method has no advantage over gradient projection for quasi-variational inequalities.

Mijajlovie et al.[25] developed a gradient projection method for solving quasivariational inequalities as:

$$\begin{aligned} s_0 &\in \mathcal{G} \\ s^{m+1} &= (1-\delta_m)s^m + \delta_m P_{N(s^m)}[s^m - \gamma M(s^m)] \end{aligned}$$

where $0 < \delta_m \leq 1$ and $\gamma > 0$ can be choosen on different ways. This method has great potential for practical applications.

In (2018), Antipin et al.[1] represented the standard gradient projection method for solving quasi-variational inequalities in the case $N(s) = k(s) + N_0$ as

$$\begin{aligned} s^{m+1} &= P_{N(s)}[s^m - \gamma M(s^m)] \\ &= k(s^m) + P_{N_0}[s^m - k(s^m) - \gamma M(s^m)] \end{aligned}$$

where $s_0$ is the initial point and $\gamma > 0$ is a parameter of the method, $N_0 \subset \mathcal{G}$ is a nonempty closed convex set in a Hilbert space $\mathcal{G}, k: \mathcal{G} \to \mathcal{G}$ is the Lipschitz continuous function, and $N: \mathcal{G} \to 2^{\mathcal{G}}$ is a set-valued mapping of the form $N(s) = k(s) + N_0,\ s \in \mathcal{G}$. They proved convergence of method under suitable conditions, with wider choice of parameters. Some well known existing methods for solving quasi-variational inequalities are found in[ 2,20,27,30,32].

Inertial-type algorithms have become increasingly popular for their convergence properties. This formulated thought is taken from the field of second order dissipative dynamical systems [3, 54. In (2019), Shehu et al. 36 developed an inertial-type algorithm with special parameters as:

$$\begin{aligned} t^m &= s^m + \Theta_m(s^m - s^{m-1}) \\ s^{m+1} &= (1-\delta_m)t^m + \delta_m P_{N(t^m)}\big(t^m - \gamma M(t^m)\big). \end{aligned}$$

and proved its strong convergence theorems. Inertial terms speed up existing algorithms, see for example,[ 6,7,11,12,37].

Motivated by research activities in this direction, we introduced a new inertial implicit iterative scheme for solving QVIs with the special choice of parametrs.

The recapitulate of the paper is designed as: we mentioned fundamental results in terms of lemmas and definitions in section 2, which we require for the core result of the paper. Iterative method with inertial effect and special choice of parameters is presented and analysed in section 3. Complexity bound of the algorithm is found in section 4 and in the last section concluion given.

## 2. Preliminaries

We take symbol $\mathcal{G}$ to represent real Hilbert space with its norm $\|.\|$ $and inner product$ $\langle\cdot,\cdot\rangle$. Consider nonlinear operator $M: \mathcal{G} \to \mathcal{G}$ and set-valued mapping $N: \mathcal{G} \to 2^{\mathcal{G}}$ which associates a closed and convex set $N(s) \subseteq \mathcal{G}$ for any element $s \in \mathcal{G}$. With this information, we consider the following QVI, for which we are finding a point $s \in N(s)$ and

$$\langle M(s), t - s\rangle \geq 0 \text{ for all } t \in N(s) \qquad \textbf{(2.1)}$$

Clearly, if $N(s) = N$ for all $s \in \mathcal{G}$, the problem reduces to the classical variational inequaity that finding $s \in N$ such that

$$\langle M(s), t - s\rangle \geq 0 \text{ for all } t \in N \qquad \textbf{(2.2)}$$

The following notations and definitions are required to prove our result.

Definition (2.1). Let $M: \mathcal{G} \to \mathcal{G}$ be a given mapping, then a mapping $M$ is called $P$-Lipschitz continuous if for any $P > 0$

$$\| M(s) - M(t) \| \leq P \| s - t \| \quad \text{for all } s, t \in \mathcal{G}$$

Definition (2.2). The mapping $M: \mathcal{G} \to \mathcal{G}$ is called u-strongly monotone, if for any $u > 0$

$$\langle M(s) - M(t), s - t\rangle \geq u \| s - t \|^2 \quad \text{for all } s, t \in \mathcal{G}$$

Definition (2.3). The mapping $M: \mathcal{G} \to \mathcal{G}$ is called monotone, if

$$\langle M(s)-M(t), s-t\rangle \geq 0 \text{ for all } s,t \in \mathcal{G}$$

Let $N$ be a nonempty, closed and convex subset of $\mathcal{G}$. For each point $s \in \mathcal{G}$, there exist a unique nearest point in $N$, denoted by $P_N(s)$, such that

$$\|s-P_N(s)\| \leq \| s-t \| \quad \text{for all } t \in N.$$

The mapping $P_N: \mathcal{G} \to N$ is called metric projection of $\mathcal{G}$ onto $N$ and is characterized by the following two properties see, e.g., [19] as:

$$P_N(s) \in N$$

and

$$\langle s-P_N(s), P_N(s)-t\rangle \geq 0 \text{ for all } s \in \mathcal{G}, t \in N \qquad \textbf{(2.3)}$$

and if $N$ is a hyperplane, then (2.2 becomes an equality.

The theory about existence of solutions differ between variational and quasivariational inequalities. For example, variational inequality has a unique solution for strongly monotonicity and Lipschitz continuity of the operator $M$ on closed and convex set. But these conditions are not sufficient for existence and uniqueness of solutions for quasi-variational inequalities. The following statement related to the existence of solutions of quasi-variational inequaliites (2.1) is valid:

**Lemma (2.1)**. [27] Let the following assumptions holds

(i) $M: \mathcal{G} \to \mathcal{G}$ be $\xi$-strongly monotone and $\tau$-Lipschitz continuous, respectively.

(ii) Also if there exists $\mathcal{N} \geq 0$

$$\|P_{N(s)}(z)-P_{N(t)}(z)\| \leq \mathcal{N} \| s-t \|, \; s,t,z \in \mathcal{G} \quad \textbf{(2.4)}$$

whereN(.) is a set-valued mapping with nonempty, closed and convex values,

(iii) $\mathcal{N}+\sqrt{1-\frac{\xi^2}{\tau^2}} < 1$

Then quasi-variational inequality (2.1) has unique solution.

If $N(s) = N_0$ is free from $s$, then we may take $\mathcal{N} = 0$ in (2.4), and hence (iii) is satisfied. In this case problem (2.1) has a unique solution if $(i)$ is satisfied, which reduces to the result for variational inequalities. The assumption (2.4) is a strengthening of the contraction property for set-valued mapping $N(s)$.In many applications the convex valued set $N(s)$ is written as $N(s) = k(s) + N_0$, where $k(s)$ is a Lipschitz continuous mapping with constant $\mathcal{N}$ and $N_0$ is a closed convex set. In this case, (2.4) holds with the same value of $\mathcal{N}$ see [26].

**Lemma (2.2).[**26] Let function $k: \mathcal{G} \to \mathcal{G}$ be Lipschitz continuous with Lipschitz constant $\mathcal{N}$ and set $N_0$ be a closed convex set. Then

$$N(s) = k(s) + N_0 \qquad \textbf{(2.5)}$$

satisfies (2.4) with the same value of $\mathcal{N}$.

**Lemma(2.3).**[27] Let $N(.) be\ a\ set-valued\ mapping\ with\ non-empty, closedand$ convex values in $\mathcal{G}$. Then $s \in N(s)$ is a solution of quasi-variational inequality (2.1) if and only if for any $\gamma > 0$ it holds that

$$s = P_{N(s)}(s-\gamma M(s)). \qquad \textbf{(2.6)}$$

**Lemma (2.4).** [13] Let $\{s_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers and let $\{v_n\}_{n=0}^{\infty}$ be a real sequence in [0,1] such that

$$\sum_{n=0}^{\infty} v_n = \infty$$

if there exists a positive integer $n_0$ such that

$$s_{n+1} \leq (1-v_n)s_n + v_n w_n, \text{ for all } n \geq n_0,$$

where $w_n \geq 0$ for all $n = 0,1,2, \dots$ and $w_n \to 0$ as $n \to \infty$, then we have

$$\lim_{n\to\infty} s_n = 0$$

## 3. Iterative Algorithm

In this section, inertial-implicit projection method with varying step sizes is introduced which establishes strong convergence theorem. Since Lemma (2.3) implies that quasi-variational inequality is equivalent to fixed point problem. With this formulation we propose an implicit iterative method for solving quasi-variational inequality.

Iterative Algorithm 3.1. Select arbitrary starting points $s^0, s^1 \in \mathcal{G}$

Iterative step: Given the iterates $s^m$ and $s^{m-1}$, compute the next iterate $s^{m+1}$ as follows:

$$\begin{aligned} t^m &= s^m + \Theta_m(s^m - s^{m-1}) \\ s^{m+1} &= (1-\delta_m)t^m + \delta_m P_{k(t^m)}\left[\frac{s^m+t^m}{2} - \rho M\left(\frac{s^m+t^m}{2}\right)\right] \end{aligned}$$

**(3.1)**

$$0 \le \Theta_m \le \Theta_m^-, \Theta_m^- = \begin{cases} \min\left\{\frac{m-1}{m+\eta-1}, \frac{\epsilon_m}{\|s^m - s^{m-1}\|}, \text{if } s^m \neq s^{m-1}\right\} \\ \frac{m-1}{m+\eta-1}, \ \text{if } s^m = s^{m-1} \end{cases} \quad \textbf{(3.2)}$$

for some $\eta \ge 3$ and $\epsilon_m \in ]0, \infty[$. We observe that in this case, algorithm generates a sequence such that $\sum_{m=1}^{\infty} \Theta_m \|s^m - s^{m-1}\| < \infty$, because for every $m \ge 1$ we get $\Theta_m \|s^m - s^{m-1}\| \le \epsilon_m$ when $s^m \neq s^{m-1}$ and $\Theta_m \|s^m - s^{m-1}\| = 0$ when $s^m = s^{m-1}$.

## 4. Main Result

**Theorem (3.1).** Consider the QVI (2.1) with $M$ being $\xi$-strongly monotone and $\tau$-Lipschitz continuous and if there exist $\mathcal{N} \ge 0$ such that (2.4) holds. Let $\{s^m\}$ be generated by algorithm (3.1) with the updating rule (3.2). In addition that for $\rho \ge 0$, the condition

$$\left|\rho - \frac{\xi}{\tau}\right| < \frac{\sqrt{\xi^2 - \rho^2 \mathcal{N}}}{\tau^2} \quad \textbf{(3.3)}$$

where $\tau = \left((1-\delta_m) + \delta_m \mathcal{N} + \delta_m \beta\right), \eta = \mathcal{N} + 2\beta, \beta = \left(\frac{1-2\rho\xi+\rho^2\tau^2}{2}\right)$, sequence $\{\delta_m\} \subseteq\ ]0,\ 1]$ satisfies $\sum_{m=1}^{\infty} \delta_m = \infty$ and $\{\epsilon_m\}$ satisfies $\sum_{m=1}^{\infty} \epsilon_m < \infty$, then $\{s^m\}$ generated by (3.1) converges strongly to the unique solution $s \in N(s)$ of the problem (2.1).

Proof. We know that

$$s = (1-\delta_m)s + \delta_m P_{N(s)}\left[\frac{s+s}{2} - \rho M\left(\frac{s+s}{2}\right)\right]$$

Now

$$\begin{aligned} \|s^{m+1} - s\| = &\ \Big\| (1-\delta_m)t^m + \delta_m P_{N(t^m)}\left[\frac{s^m+t^m}{2} - \rho M\left(\frac{s^m+t^m}{2}\right)\right] - \\ &\left[(1-\delta_m)s + \delta_m P_{N(s)}\left(\frac{s+s}{2} - \rho M\left(\frac{s+s}{2}\right)\right)\right] \Big\| \\ \le &\ \|(1-\delta_m)(t^m - s)\| + \\ &\delta_m \Big\| P_{N(t^m)}\left[\frac{s^m+t^m}{2} - \rho M\left(\frac{s^m+t^m}{2}\right)\right] - \\ &P_{N(s)}\left[\frac{s+s}{2} - \rho M\left(\frac{s+s}{2} - \rho M\left(\frac{s+s}{2}\right)\right)\right] \\ \le &\ (1-\delta_m)\|t^m - s\| + \\ &\delta_m \Big\| P_{N(t^m)}\left[\frac{s^m+t^m}{2} - \rho M\left(\frac{s^m+t^m}{2}\right)\right] - \\ &P_{k(s)}\left[\frac{s^m+t^m}{2} - \rho M\left(\frac{s^m+t^m}{2}\right)\right] \Big\| + \delta_m \\ &\left\| P_{N(s)}\left[\frac{s^m+t^m}{2} - \rho M\left(\frac{s^m+t^m}{2}\right)\right] - P_{N(s)}\left[\frac{s+s}{2} - \rho M\left(\frac{s+s}{2}\right)\right] \right\| \end{aligned}$$

$$\begin{aligned} \le &\ (1-\delta_m)\|t^m - s\| + \\ &\delta_m \mathcal{N}\|t^m - s\| + \delta_m \Big\| \frac{s^m+t^m}{2} - \frac{s+s}{2} \\ &-\rho\left(M\left(\frac{s^m+t^m}{2}\right) - M\left(\frac{s+s}{2}\right)\right) \Big\| \end{aligned} \quad \textbf{(3.4)}$$

Now, since $M$ is $\xi$-strongly monotone and $\tau$-Lipschitz continuous, we have

$$\begin{aligned} &\left\| \frac{s^m+t^m}{2} - \frac{s+s}{2} - \rho\left(M\left(\frac{s^m+t^m}{2}\right) - M\left(\frac{s+s}{2}\right)\right) \right\|^2 \\ &= \left\|\frac{s^m+t^m}{2} - \frac{s+s}{2}\right\|^2 - \\ &2\rho\left\langle M\left(\frac{s^m+t^m}{2}\right) - M\left(\frac{s+s}{2}\right), \frac{s^m+t^m}{2} - \frac{s+s}{2}\right\rangle \\ &\quad +\rho^2\left\|M\left(\frac{s^m+t^m}{2}\right) - M\left(\frac{s+s}{2}\right)\right\|^2 \\ &\le \left\|\frac{s^m+t^m}{2} - \frac{s+s}{2}\right\|^2 - 2\rho\xi\left\|\frac{s^m+t^m}{2} - \frac{s+s}{2}\right\|^2 \\ &+\rho^2\tau^2\left\|\frac{s^m+t^m}{2} - \frac{s+s}{2}\right\|^2 \\ &= (1 - 2\rho\xi + \rho^2\tau^2)\left\|\frac{s^m+t^m}{2} - \frac{s+s}{2}\right\| \end{aligned} \quad \textbf{(3.5)}$$

Now

$$\left\|\frac{s^m+t^m}{2} - \frac{s+s}{2}\right\| \le \frac{1}{2}\|s^m - s\| + \frac{1}{2}\|t^m - s\| \textbf{(3.6)}$$

Using (3.6) in (3.5), we have

$$\left\| \frac{s^m + t^m}{2} - \frac{s+s}{2} - \rho\left(M\left(\frac{s^m + t^m}{2}\right) - M\left(\frac{s+s}{2}\right)\right) \right\|^2$$
$$\leq (1 - 2\rho\xi + \rho^2\tau^2)\left[\frac{1}{2}\|s^m - s\| + \frac{1}{2}\|t^m - s\|\right]$$
$$= \beta[\|s^m - s\| + \|t^m - s\|] \qquad (\mathbf{3.7})$$

where $\beta = \frac{(1-2\rho\xi+\rho^2\tau^2)}{2}$

Using (3.7) in (3.4), we have

$$\|s^{m+1} - s\| \leq \big((1-\delta_m) + \delta_m\mathcal{N} + \delta_m\beta\big)\|t^m - s\| + \delta_m\beta\|s^m - s\| \qquad \textbf{(3.8)}$$

Now

$$\|t^m - s\| = \|s^m + \Theta_m(s^m - s^{m-1}) - s\| \leq \|s^m - s\| + \Theta_m\|s^m - s^{m-1}\| \quad (\mathbf{3.9})$$

Using $(\mathbf{3.9})$ in $(\mathbf{3.8})$**,** we obtain

$$\|s^{m+1} - s\| \leq \big((1-\delta_m) + \delta_m\mathcal{N} + \delta_m\beta\big)(\|s^m - s\| + \Theta_m\|s^m - s^{m-1}\|) + \delta_m\beta\|s^m - s\|$$
$$= [(1-\delta_m) + \delta_m\mathcal{N} + 2\delta_m\beta]\|s^m - s\| + \big((1-\delta_m) + \delta_m\mathcal{N} + \delta_m\beta\big)\Theta_m\|s^m - s^{m-1}\|$$
$$= [(1-\delta_m) + \delta_m(\mathcal{N} + 2\beta)]\|s^m - s\| + \zeta_n\Theta_m\|s^m - s^{m-1}\|$$
$$= [(1 - \delta_m(1 - (\mathcal{N} + 2\beta))]\|s^m - s\| + \zeta_n\Theta_m\|s^m - s^{m-1}\|$$
$$= [(1 - \delta_m(1 - \omega))]\|s^m - s\| + \zeta_n\Theta_m\|s^m - s^{m-1}\| \ (\mathbf{3.10})$$

where $\zeta_n = \big((1-\delta_m) + \delta_m\mathcal{N} + \delta_m\beta\big), \omega = \mathcal{N} + 2\beta.$

observe that by condition **(3.3),** we have $0 < \omega < 1$, since $\sum_{m=1}^{\infty}\Theta_m\|s^m - s^{m-1}\| < \infty$, using Lemma(2.4), we get $s^m \to s$, as $m \to \infty$.

**Remark (3.1).** However our Theorem (3.1) still holds if in (3.2) the term $\frac{m-1}{m+\delta-1}$ is replaced with some constant in [0,1[. The idea of using such inertial term was actually introduced in [8,10] and interest for taking $\eta \geq 3$ lies in the fact that was actually used by Attouch and Peypouquet [8] and Attouch et al. [7], in which they proved the fast convergence for this hypothesis.

In the next section we present the complexity bound of Algorithm(3.1) with the updating rule (3.2 ).

## 5. Complexity bound of the algorithm

**Theorem (4.1).** Consider the QVI (2.1) with the same assumptions as in theorem (3.1) above. Let $\{s^m\}$ be generated by (3.1) with the updating rule (3.2) and let $s \in N(s)$ be the unique solution of the QVI(2.1) . Let $\delta_m = \delta$ and $\epsilon_m = \epsilon$ be constant. Then for any $\chi \in ]0, \delta(1 - (\mathcal{N} + 2\beta)[$, for any

$$m \geq \bar{m} = \left[\log_{(1-\chi)}\left(\left(\frac{\epsilon}{\|s^0 - s\|}\right)\left(\frac{1-\delta(1-(\mathcal{N}+\beta))}{\delta(1-(\mathcal{N}+2\beta))-\chi}\right)\right)\right] \qquad \textbf{(4.1)}$$

assuming $\bar{m} \geq 0$, it holds that

$$\|s^m - s\| \leq \left[\frac{1-\delta(1-(\mathcal{N}+\beta))}{\delta(1-(\mathcal{N}+2\beta))-\chi} + (1 - \delta\big(1 - (\mathcal{N} + \beta)\big)\right]\epsilon \qquad \textbf{(4.2)}$$

Proof. From the proof of the theorem (3.1) above, for any $m \geq 1$, we get

$$\|s^{m+1} - s\| \leq (1 - \delta(1 - (\mathcal{N} + 2\beta)))\|s^m - s\| + (1 - \delta(1 - (\mathcal{N} + \beta)))\Theta_m\|s^m - s^{m-1}\|$$
$$\leq \Big(1 - \delta\big(1 - (\mathcal{N} + 2\beta)\big)\Big)\|s^m - s\| + \Big(1 - \delta\big(1 - (\mathcal{N} + \beta)\big)\Big)\epsilon \qquad (\mathbf{4.3})$$

because, $(1 - \delta(1 - (\mathcal{N} + 2\beta))) \geq 0$, without loss of generality, assume that for $m < \bar{m}$, we get

$$\|s^m - s\| \geq \epsilon\frac{(1-\delta(1-(\mathcal{N}+\beta)))}{\delta(1-(\mathcal{N}+2\beta))-\chi} \qquad \textbf{(4.4)}$$

from (4.3 )and (4.4), we obtain for every $m < \bar{m}$

$$\|s^{m+1} - s\| \leq (1-\chi)\|s^m - s\| \qquad \textbf{(4.5)}$$

therefore by definition of $\bar{m}$, it holds that

$$\|s^{\bar{m}} - s\| \leq (1-\chi)^{\bar{m}}\|s^0 - s\| \leq \epsilon\frac{1 - \delta(1 - (\mathcal{N} + \beta))}{\delta(1 - (\mathcal{N} + 2\beta)) - \chi}$$

for any $m > \bar{m}$, there are two possiblities if

$$\|s^{m-1} - s\| \leq \epsilon\frac{1 - \delta(1 - (\mathcal{N} + \beta))}{\delta(1 - (\mathcal{N} + 2\beta)) - \chi}$$

then by (4.3) and recalling that

$(1 - \delta(1 - (\mathcal{N} + 2\beta))) \leq 1$, we get

$$\|s^m - s\| \leq \left[\frac{1-\delta(1-(\mathcal{N}+\beta))}{\delta(1-(\mathcal{N}+2\beta))-\chi} + (1-\delta(1 - (\mathcal{N}+\beta)))\right]\epsilon$$

otherwise if

$$\frac{1-\delta(1-(\mathcal{N}+\beta))}{\delta(1-(\mathcal{N}+2\beta))-\chi} \leq \|s^{m-1} - s\| \leq \epsilon\left(\frac{1-\delta(1-\delta(\mathcal{N}+\beta))}{\delta(1-(\mathcal{N}+2\beta))-\chi} + (1-\delta(1-(\mathcal{N}+\beta)))\right.$$

then,

$$\|s^m - s\| \leq (1-\chi)\|s^{m-1} - s\| \leq \|s^{m-1} - s\|$$

and hence the desired result holds.

## 6. CONCLUSION

In this paper, we proposed an implicit projection method for solving QVIs in real Hilbert spaces. We proved convergence of the inertial implicit projection method under suitable conditions. The complexity bound of an algorithm determines how fast it will run and how much memory it will require, we have found the complexity bound of the proposed algorithm. Researchers can use Noor's techniques [33] to analyze quasi-variational inequalities using error estimates and sensitivity analysis as well.

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